# Evaluating the Delay Robustness of Interconnected Passive Systems with a Frequency-Dependent Integral Quadratic Constraint

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***Abstract*— We consider interconnections of output strictly passive systems and study their robustness in the presence of delay. We establish a new integral quadratic constraint (IQC), which captures the magnitude roll-off at high frequencies. The benefit of the roll-off IQC is exemplified in a study of the stability of a cyclic interconnection structure. When used in conjunction with the IQC describing passivity, we obtain a stability bound on the gain that depends on the size of the delay and relaxes a small-gain bound.**

## I. Introduction

A series of recent publications presented a *passivity* approach for overcoming the complexity of high-order differential equation models arising in communication networks [17], [8], [1], cooperative robotic vehicles [2], [11], [5], and biochemical reaction networks [3], [4]. This approach exploits the structure of the network and breaks up the design and analysis procedures into two levels: At the network level, one represents the components with passivity properties as abstractions of their complex dynamic models. At the component level, one studies the individual dynamic models and verifies or assigns passivity properties without relying on further knowledge of the network.

Passivity is a physically relevant property that is inherent in the aforementioned applications and avoids the conservatism of stability conditions obtained with a small-gain approach. However, in the presence of time delay, passivity properties alone do not guarantee robustness, and additional properties must be employed. In contrast, a small-gain approach ensures robustness to delay; however, it does not take into account the duration of the delay, and may lead to conservative criteria.

The purpose of this paper is to derive stability conditions that converge to the passivity estimates as the duration of delay approaches zero, and to the small-gain estimates as the amount of delay approaches infinity. We follow the *integral quadratic constraint* (IQC) framework [13], [12] for stability of interconnections, and employ two IQCs simultaneously: The first one is an *output strict passivity* IQC that describes the gain and phase properties of the components comprising the network, and the second one is a "roll-off" IQC that is frequency-dependent, and carries information about the time scales of these components. Indeed, the destabilizing effect of delay depends on its magnitude relative to the time scales of the system and, thus, the second IQC is essential for obtaining stability estimates that are sensitive to the amount of delay.

For a concrete demonstration of the advantage of the roll-off IQC, we study a *cyclic* interconnection structure for which a stability bound was derived in [3] using output strict passivity properties of the components and assuming no delay. This bound is referred to as the "secant criterion," and has the form $\gamma \cos^n(\pi/n) < 1$ where $n$ is the number of blocks in the feedback loop, and $\gamma$ is the product of their gains. As an illustration, for $n = 3$ blocks, the secant criterion restricts the gain by $\gamma < \sec^3(\pi/3) = 8$.

We first show that, in the presence of delay $T$, an application of the IQC stability theorem using only the output strict passivity IQC yields the small-gain condition $\gamma < 1$ regardless of the value of $T$. By including the roll-off IQC, we drive a new stability test in which the admissible gain $\gamma$ is a function of the delay $T$ and the dimension of the system $n$. This function converges to 1 as $T \to \infty$ and to the secant condition as $T \to 0$, as desired.

Section III reviews the output strict passivity IQC and the IQC stability theorem. Section IV defines the roll-off IQC. Section V studies cyclic interconnections, presents the results of the IQC approach, and states the theorem. Section VI proves the main theorem. Finally, Section VII presents the conclusions.

## II. Notation

Let $\mathbb{N}$ be the set of natural numbers. Let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers. The set of $m \times n$ matrices whose elements are in $\mathbb{R}$ or $\mathbb{C}$ are denoted as $\mathbb{R}^{m\times n}$ and $\mathbb{C}^{m\times n}$. A single superscript index denotes vectors, e.g. $\mathbb{R}^m$ is the set of $m \times 1$ vectors whose elements are in $\mathbb{R}$.

Let $e_i \in \mathbb{R}^n$ be the unit vector with zeros everywhere except in the *ith* element.

$\mathbf{L}_2^m$ is the space of $\mathbb{R}^m$-valued functions $f : [0, \infty) \to \mathbb{R}^m$ of finite energy $\|f\|^2 = \int_0^\infty f(t)^T f(t)\, dt$.

Let $\Pi : j\mathbb{R} \to \mathbb{C}^{(l+m)\times(l+m)}$ be a measurable, bounded Hermitian-valued function. A bounded, causal operator $\Delta$ mapping $\mathbf{L}_{2e} \to \mathbf{L}_{2e}$ is said to satisfy *the IQC defined by* $\Pi$, if for all $v \in \mathbf{L}_2$, with $y = \Delta(v)$, the inequality

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{y}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{v}(j\omega) \\ \widehat{y}(j\omega) \end{bmatrix} d\omega \geq 0 \qquad (1)$$

holds.

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## III. Stability of Interconnected OSP Systems

Consider Figure 1, where $G$ has the form:

$$G(s) = \sum_{i=1}^{m} G_i e^{sT_i} + G_0, \tag{2}$$

where $G_i$ for $i = 0, \dots, m$ are proper, rational functions without poles in the closed right-half plane[1], and each $\Delta_i$ is a bounded, causal operator. Our main interest is in the situation where $\Delta_i$ are dynamical blocks representing the components of a network, and $G$ is a matrix representing their interconnection structure.

Suppose each $\Delta_i$ satisfies the IQC

$$\Pi_1 = \begin{bmatrix} 0 & 0.5 \\ 0.5 & -1 \end{bmatrix}, \tag{3}$$

which describes an *output strict passivity* (OSP) property with gain 1. We choose the gain to be 1 without loss of generality, since we can modify $G(s)$ to absorb a different value of the gain.

Next we define

$$\Pi(j\omega) = \sum_{i=1}^{n} \alpha_i \Pi_1 \otimes e_i e_i^T, \ \alpha_i \in \mathbb{R}, \tag{4}$$

and note that the block diagonal concatenation $\Delta := \mathrm{diag}(\Delta_i)$ satisfies the IQC defined by (4) for any choice of $\alpha_i \geq 0$, $i = 1, \dots, n$.

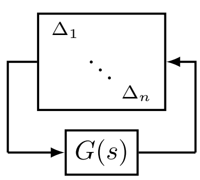


Fig. 1. Feedback Interconnection of $\Delta$ and $G$

From this point on, we assume that the following conditions hold as stipulated in [13]:

1) For every $\tau \in [0, 1]$, the interconnection of $G$ and $\tau\Delta$ is well-posed,
2) For every $\tau \in [0, 1]$, the IQC defined by $\Pi$ is satisfied by $\tau\Delta$.

*Theorem 1:* If there exists $\epsilon > 0$, $P \in \mathbb{R}^{n\times n}$, $P \succ 0$, such that

$$P(G(j\omega) - I) + (G(j\omega) - I)^* P \preceq -2\epsilon I \ \forall \omega \in \mathbb{R}, \tag{5}$$

then the feedback interconnection of $G$ and $\Delta$ is stable. □

*Proof:* We show that the theorem is equivalent to the third (and final) condition of the IQC stability theorem in [13]: If $\exists \epsilon > 0$ and $\alpha_i > 0$ with $\Pi$ as in (4) such that

$$H_\Pi := \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \preceq -\epsilon I \ \forall \omega \in \mathbb{R} \tag{6}$$

holds, then the interconnection of $G$ and $\Delta$ is stable.

To show this, we let $P := \mathrm{diag}(\alpha_i) \succ 0$ and note that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} = \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* (\Pi_1 \otimes P) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} = \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} 0 & 0.5P \\ 0.5P & -P \end{bmatrix} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}.$$

It then follows that condition (6) is equivalent to (5). ■

## IV. Roll-off IQC

In order to evaluate the delay robustness of OSP systems interconnected as in Figure 1, we now introduce a roll-off IQC, which describes a reduction in the gain with increasing frequency. This IQC will be particularly useful when $G$ contains a delay element, since the roll-off characterizes the time-constants within the $\Delta_i$ blocks. The roll-off IQC has the form:

$$\Pi_{2\tau}(j\omega) := \begin{bmatrix} 1 & 0 \\ 0 & \frac{-1-(\frac{\omega}{\omega_c})^2}{1+\tau(\frac{\omega}{\omega_c})^2} \end{bmatrix},$$

where $\omega_c$ is the "corner frequency" and $0 < \tau \ll 1$ is introduced to render $\Pi_{2\tau}(j\omega)$ proper. Since the time variable can be scaled appropriately, in the rest of the paper we normalize $\omega_c$ to $\omega_c = 1$ and, instead, vary the magnitude of the time delay. In addition, for the analysis, we will consider $\Pi_{2\tau}$ at $\tau = 0$, which we define as

$$\Pi_2(j\omega) := \begin{bmatrix} 1 & 0 \\ 0 & -(1+\omega^2) \end{bmatrix}.$$

In this paper, we propose the joint use of the OSP IQC $\Pi_1$ and the roll-off IQC $\Pi_2$ when the interconnection, $G(s)$, contains delay elements. Since $\Pi_1$ is a frequency-independent IQC, the time-scales of an operator which satisfies $\Pi_1$ are not constrained. Likewise, the IQC $\Pi_2$ does not constrain the phase properties of operators. Thus, we select the combined IQC

$$\Pi(j\omega) := \sum_{i=1}^{n} (\alpha_{i1}\Pi_1 + \alpha_{i2}\Pi_2(j\omega)) \otimes e_i e_i^T \tag{7}$$

and search for $\alpha_{i1}, \alpha_{i2} \geq 0$ such that (6) holds with the form in (7). We refer to $\Pi^\tau$ as the combined IQC (7) when $\Pi_2$ is replaced with $\Pi_{2\tau}$.

The following lemma proves that if $H_\Pi \preceq -\epsilon I$ holds on a special compact set of $\omega$, then $H_\Pi \preceq -\epsilon I$ for all $\omega \in \mathbb{R}$. We further show in Lemma 2 that we can replace $\Pi_{2\tau}$ with $\Pi_2$, which yields a cleaner analysis in the proof of the main theorem in Section VI.

*Lemma 1:* Suppose $G_i$ in (2) are constant for $i = 0, \dots, m$ and all of the delays $T_i$ are commensurate[2] so that there exists $\hat{T}$ such that $T_i = N_i \hat{T}$ for all $i = 1, \dots, m$ and some $N_i \in \mathbb{N}$. Let $H_\Pi$ be defined as in (6) with $\Pi$ in (7) . Then, there exists an $\epsilon > 0$ such that $H_\Pi \preceq -\epsilon I$ holds for all $\omega \in \mathbb{R}$ if and only if $H_\Pi \preceq -\epsilon I$ holds for $\omega \in \left[-\frac{\pi}{\hat{T}}, \frac{\pi}{\hat{T}}\right]$.

[1]The authors in [13] consider a rational $G$, but the general form in (2) is admissible, as alluded to in [12].

[2]If all of the ratios between delays $\frac{T_i}{T_j}$ for $i, j = 1, \dots, m$ are rational numbers, then the delays are said to be *commensurate* [10].

*Proof:* Since $G(j\omega)$ is periodic and the lower right $n\times n$ block of $\Pi$ is even and decreasing (in a semidefinite ordering) with increasing $\omega \in [0,\infty)$, $\lambda_{\max}(H_\Pi)$ will achieve a maximum on $\omega \in \left[-\frac{\pi}{T}, \frac{\pi}{T}\right]$. ■

*Lemma 2:* Suppose the conditions in Lemma 1 hold. Let $\mathcal{T} = \left[-\frac{\pi}{T}, \frac{\pi}{T}\right]$. If there exists $\epsilon > 0$, $\alpha_{i1}, \alpha_{i2} \geq 0$, such that $H_\Pi \preceq -\epsilon I$ holds for all $\omega \in \mathcal{T}$, then there exists a $\tau > 0, \tilde{\epsilon} > 0$, $\tilde{\alpha}_{i1}, \tilde{\alpha}_{i2} \geq 0$ such that

$$H_{\Pi^\tau} \preceq -\tilde{\epsilon} I \tag{8}$$

holds for all $\omega \in \mathcal{T}$.

*Proof:* Assume there exists $\epsilon > 0, \alpha_{i1} \geq 0, \alpha_{i2} \geq 0$ such that $H_\Pi \preceq -\epsilon I$ for all $\omega \in \mathcal{T}$. Denote $\Pi^\tau(2,2)$ and $\Pi(2,2)$ as the lower right $n \times n$ blocks of $\Pi^\tau$ and $\Pi$. Let $\tilde{\alpha}_{i1} := \alpha_{i1}, \tilde{\alpha}_{i2} := \alpha_{i2}$, and let

$$\hat{\alpha} = \max_i \ \ \tilde{\alpha}_{i2}. \tag{9}$$

For $\omega \in \{\mathcal{T} \setminus 0\}$, if

$$\tau < \frac{\lambda_{\max}\left(-H_\Pi\right)}{\hat{\alpha}\tau \left(\frac{\pi}{T}\right)^2 \left(1 + \left(\frac{\pi}{T}\right)^2\right)}, \tag{10}$$

then

$$\overline{\sigma}\left(\Pi^\tau(2,2) - \Pi(2,2)\right) < \lambda_{\max}\left(-H_\Pi\right). \tag{11}$$

Note that for any $\tau$,

$$H_\Pi^\tau = H_\Pi + \Pi^\tau(2,2) - \Pi(2,2). \tag{12}$$

Since (11) holds, the negativity of $H_{\Pi^\tau}$ is preserved for $\omega \in \{\mathcal{T} \setminus 0\}$. At $\omega = 0$, the negativity is preserved since $H_{\Pi^\tau}(0) = H_\Pi(0)$. Hence, there exists an $\tilde{\epsilon}$ such that (8) holds for $\omega \in \mathcal{T}$. ■

## V. Cyclic Interconnections with Delay

In this section we make the advantage of the combined IQC (7) explicit by studying a special interconnection structure whose stability properties in the absence of delay are characterized in [3]. Let $G(j\omega)$ be of the form

$$G(j\omega) = \begin{bmatrix} 0 & 0 & \cdots & 0 & g_1(j\omega) \\ g_2(j\omega) & 0 & 0 & & 0 \\ 0 & g_3(j\omega) & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & g_n(j\omega) & 0 \end{bmatrix}, \tag{13}$$

where

$$g_i(j\omega) = \rho_i e^{-j\beta_i(\omega)}, \ \rho_i > 0, \tag{14}$$

such that

$$\prod_{i=1}^n g_i(j\omega) = \gamma e^{j(\pi - \omega T)}. \tag{15}$$

Figure 2 represents the interconnection of Figure 1 with $G$ defined as in (13). The phase condition in (15) means that this is a negative feedback loop with delay $T$. Now, we prove that the result of the IQC based analysis depends only on the total gain $\gamma$ and total delay $T$, and not the particular choice of each $\rho_i$ and $\beta_i(\omega)$.

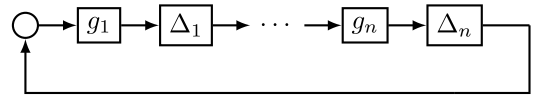


Fig. 2. Interconnection of $\Delta$ and cyclic $G$

*Theorem 2:* Let $G(j\omega)$ and $\tilde{G}(j\omega)$ represent two different cyclic matrices as in (13), each with different choice of values for the $g_i(j\omega)$ such that both matrices satisfy (15) with a common $\gamma$ and $T$. Let $g_i(j\omega)$ and $\tilde{g}_i(j\omega)$ indicate the particular choice for $G(j\omega)$ and $\tilde{G}(j\omega)$. Let $\{\Pi_k(j\omega)\}_{k=1}^p$ represent an arbitrary set of IQCs.

Define

$$\Pi(j\omega) = \sum_{i=1}^n \sum_{k=1}^p \alpha_{ik} \Pi_k(j\omega) \otimes e_i e_i^T$$

and

$$\tilde{\Pi}(j\omega) = \sum_{i=1}^n \sum_{k=1}^p \tilde{\alpha}_{ik} \Pi_k(j\omega) \otimes e_i e_i^T.$$

There exist constants $\alpha_{ik} \geq 0$ and $\epsilon > 0$ such that for all $\omega \in \mathbb{R}$

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \preceq -\epsilon I, \tag{16}$$

if and only there exist constant $\tilde{\alpha}_{ik} \geq 0$ and $\tilde{\epsilon} > 0$ such that for all $\omega \in \mathbb{R}$

$$\begin{bmatrix} \tilde{G}(j\omega) \\ I \end{bmatrix}^* \tilde{\Pi}(j\omega) \begin{bmatrix} \tilde{G}(j\omega) \\ I \end{bmatrix} \preceq -\tilde{\epsilon} I \tag{17}$$

□

*Proof:* See Appendix. ■

Now we will study how delay affects the stability of the cyclic interconnection in (13). We first consider the case $T = 0$ and recall the following stability test from [3]:

*Theorem 3:* Suppose each $\Delta_i$ satisfies the IQC defined by $\Pi_1$. Then the feedback interconnection of $\Delta$ and $G$ is stable if

$$\gamma \cos\left(\frac{\pi}{n}\right)^n < 1. \tag{18}$$

□

Although [3] did not use the IQC formulation, the stability criterion was identical to (5) with $G$ as in (13)-(15) and $T = 0$. The existence of a diagonal $P \succ 0$ was shown in [3, Theorem 1] to be equivalent to (18).

Now we consider the case $T \neq 0$, and show that employing the IQC $\Pi_1$ alone yields a conservative result that is independent of the size of the delay.

*Theorem 4:* Suppose $T > 0$ and each $\Delta_i$ satisfies the IQC defined by $\Pi_1$. There exists a diagonal $P \succ 0$ and $\epsilon > 0$ such that (6) with $\Pi = \Pi_1$ holds if and only if $\gamma < 1$.

□

*Proof:* From Theorem 2 , we choose $G(j\omega)$ such that

$$g_1(j\omega) = -\gamma e^{-j\omega T} \text{ and } g_i(j\omega) = 1, i \geq 2$$

without loss of generality.

($\Rightarrow$ Contradiction) Suppose $P \succ 0$ exists and $\gamma \geq 1$. At $\bar{\omega} = \frac{\pi}{T}$, $G(j\bar{\omega}) - I$ is a Metzler matrix of the form:

$$G(j\bar{\omega}) - I = \begin{bmatrix} -1 & 0 & \cdots & \gamma \\ 1 & -1 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & 0 & 1 & -1 \end{bmatrix}. \tag{19}$$

From [7, Theorem 2.3], there exists an $\epsilon > 0$ and $P \succ 0$ such that $P[G(j\bar{\omega}) - I] + [G(j\bar{\omega}) - I]^* P \preceq -\epsilon I$ if and only if the principal minors of $-(G(j\bar{\omega} - I))$ are all positive. All principal minors of $-(G(j\bar{\omega} - I))$, except the minor which is the determinant of the matrix itself, are 1. The remaining principal minor, which is

$$\begin{aligned} \det(G(j\bar{\omega}) - I) &= 1 - \gamma(-1)^{n+1}(-1)^{n-1} \\ &= 1 - \gamma(-1)^{2n} = 1 - \gamma, \end{aligned}$$

is positive only if $\gamma < 1$. Hence, when $\gamma \geq 1$ at $\omega = \bar{\omega}$, there does not exist a $P$ such that (7) holds.

($\Leftarrow$) Assume $\gamma < 1$. We will first show that a $P$ exists at $\bar{\omega} = \frac{\pi}{T}$, and then show that this $P$ can be used for any value of $\omega$. At $\omega = \bar{\omega}$, $G(j\bar{\omega}) - I$ is a Metzler matrix of the form in (19). Hence, since $\gamma < 1$, there exists a diagonal, positive definite $P$ and $\epsilon > 0$ such that

$$x^*[P(G(j\bar{\omega}) - I) + (G(j\bar{\omega}) - I)^* P]x \leq -\epsilon |x|^2 \tag{20}$$

holds for all $x \in \mathbb{C}^n$. Expanding (20) yields the condition

$$\begin{aligned} -p_1|x_1|^2 - \ldots - p_n|x_n|^2 + 0.5p_2(x_1^* x_2 + x_2^* x_1) \\ + \ldots + 0.5p_n(x_{n-1}^* x_n + x_n^* x_{n-1}) \\ +0.5\gamma p_1(x_1^* x_n + x_n^* x_1) \leq -\epsilon |x|^2. \end{aligned} \tag{21}$$

Since (21) holds for all $x \in \mathbb{C}^n$, it also holds for all $x_i = |y_i|$ for $y \in \mathbb{C}^n$. Hence, it is clear that (21) implies that

$$\begin{aligned} -p_1|y_1|^2 - \ldots - p_n|y_n|^2 + \ldots + p_2|y_1||y_2| \\ + \ldots + p_n|y_{n-1}||y_n| + \gamma p_1|y_1||y_n| \leq -\epsilon |y|^2 \end{aligned} \tag{22}$$

holds $\forall y \in \mathbb{C}^n$.

Now we consider $y^*[P(G(j\omega) - I) + (G(j\omega) - I)^* P]y$, over all $\omega \in \mathbb{R}$. Expanding $y^*[P(G(j\omega) - I) + (G(j\omega) - I)^* P]y$ yields

$$\begin{aligned} -p_1|y_1|^2 - \ldots - p_n|y_n|^2 + \ldots + 0.5p_2(y_1^* y_2 + y_2^* y_1) \\ + \ldots + 0.5p_n(y_{n-1}^* y_n + y_n^* y_{n-1}) \\ -0.5\gamma p_1(e^{-j\omega T} y_1^* y_n + e^{j\omega T} y_n^* y_1), \end{aligned}$$

which is upper bounded by (22). Hence, there exists a positive definite $P = \mathrm{diag}(p_1, \ldots, p_n)$ and $\epsilon > 0$ such that (6) holds for all $\omega \in \mathbb{R}$. ■

Since the condition $\gamma < 1$ in Theorem 4 does not depend on the duration of the delay, we conclude that using $\Pi_1$ alone leads to a conservative result.

We now present a theorem which shows that the IQC LMI stability test for a OSP system with roll-off is equivalent to a scalar test in which the admissible gain $\gamma$ is a function of the delay $T$ and the dimension of the system $n$. We provide a proof in section VI.

*Theorem 5:* For any $\gamma > 1$, $T \geq 0$, there exists $\epsilon > 0$ and $\alpha_{1i}, \alpha_{2i} \geq 0$ such that $H_\Pi \preceq -\epsilon I$ holds for all $\omega \in \mathbb{R}$ if

$$T < \frac{\pi - n \arctan\left(\sqrt{\gamma^{\frac{2}{n}} - 1}\right)}{\sqrt{\gamma^{\frac{2}{n}} - 1}}. \tag{23}$$

□

The condition in (23) is the time delay margin for a cascade of identical linear systems $\Delta_i = \frac{1}{(s+1)}$ for $i = 1, \ldots, n$, in feedback with gain $\gamma$. This is a particular system that satisfies the IQCs defined by $\Pi_1$ and $\Pi_2$, which means that the bound is tight and cannot be relaxed without further assumptions.

Notice in condition (23) that for $T = 0$, the secant condition in Theorem 3 is recovered, and that as $T \to \infty$, the small gain condition in Theorem 4 is recovered.

As a numerical example, we performed the robustness test $H_\Pi \preceq -\epsilon I$ for $n = 3$ by gridding the frequency $\omega \in \left[-\frac{\pi}{T}, \frac{\pi}{T}\right]$ with 300 points, which is plotted in Figure 3 for $n = 3$. The analytical formula in (23) is overlaid with the circles and matches the numerical test.

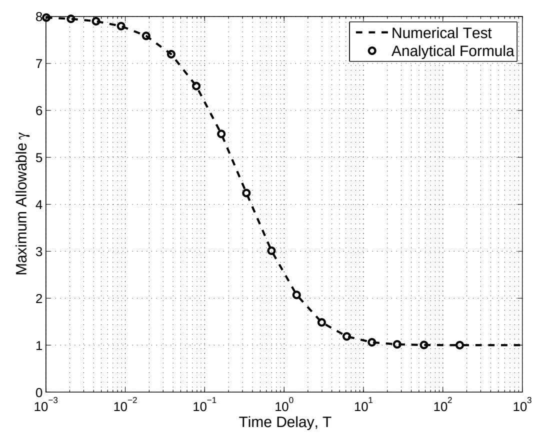


Fig. 3. A stability bound on $\gamma$ as a function of delay $T$ for $n = 3$, obtained numerically with the combined IQC. The result of the condition in (23) is superimposed.

## VI. Proof of Theorem 5

We choose $G(j\omega)$ such that

$$g_i(j\omega) := g(j\omega) \triangleq \gamma^{\frac{1}{n}} e^{j\left(-\frac{\omega T}{n} + \frac{\pi}{n}\right)}, \quad i = 1, \ldots, n. \tag{24}$$

With $\alpha_{i1} = 1$ and $\alpha_{i2} = \alpha$, $i = 1, \cdots, n$, the IQC stability condition $H_\Pi \preceq -\epsilon I$ can be written as:

$$\begin{aligned} \frac{1}{2}(G(j\omega) - I) + \frac{1}{2}(G(j\omega) - I)^* \\ +\alpha(|g(j\omega)|^2 - (1 + \omega^2)))I \preceq -\epsilon I \quad \forall \omega. \end{aligned} \tag{25}$$

Let $\mathcal{T} := \left[-\frac{\pi}{T}, \frac{\pi}{T}\right]$. From Lemma 1 , the inequality (25) need only hold for $\forall \omega \in \mathcal{T}$.

Since

$$(G(j\omega) - I) + (G(j\omega) - I)^* \tag{26}$$

is a circulant matrix, it's eigenvectors are [9]:

$$v_k = [1 \; e^{-j(k-1)\frac{2\pi}{n}} \; e^{-j(k-1)2\frac{2\pi}{n}} \; \cdots \; e^{-j(k-1)(n-1)\frac{2\pi}{n}}]^T \tag{27}$$

for $k = 1, \ldots n$ and, thus, the eigenvalues are the discrete Fourier transform coefficients of the first row which, for (26), are:

$$\lambda_k(j\omega) = -2 + g(j\omega)^* e^{-j(k-1)\frac{2\pi}{n}} + g(j\omega) e^{-j(k-1)(n-1)\frac{2\pi}{n}} \tag{28}$$

for $k = 1, \ldots n$.

Defining the matrix $V = [v_1 \cdots v_n]$ and noting that $V^{-1} = \frac{1}{n}V^*$, we conclude:

$$\frac{1}{n}V^*[(G(j\omega) - I) + (G(j\omega) - I)^*]V = \mathrm{diag}(\lambda_1(j\omega), \ldots, \lambda_n(j\omega)).$$

Multiplying (25) from the right by $V$ and $\frac{1}{n}V^*$ from the left yields a diagonal matrix. Thus, the condition in (25) becomes

$$\frac{1}{2}\lambda_k(j\omega) + \alpha(\gamma^{2/n} - (1 + \omega^2)) \leq -\epsilon, \quad k = 1, \cdots, n \quad \forall \omega \in \mathcal{T}. \tag{29}$$

Substituting (24) in (28) and simplifying yields

$$\lambda_k(j\omega) = -2 + 2\gamma^{1/n}\cos\left(\frac{\pi}{n} + (k-1)\frac{2\pi}{n} - \frac{\omega T}{n}\right) \tag{30}$$

for $k = 1, \cdots, n$. We rewrite (29) as:

$$h(\omega, k) := -1 + \gamma^{1/n}\cos\left(\frac{\pi}{n} + (k-1)\frac{2\pi}{n} - \frac{\omega T}{n}\right) + \alpha(\gamma^{2/n} - (1 + \omega^2))) \leq -\epsilon \quad k = 1, \cdots, n, \forall \omega \in \mathcal{T}.$$

Note that for $\omega \in \mathcal{T}$, $\frac{\omega T}{n} \in \left[-\frac{\pi}{n}, \frac{\pi}{n}\right]$. For $\omega \in \left[0, \frac{\pi}{T}\right]$, $h(\omega, k)$ has the largest value when $k = 1$. For $\omega \in \left[\frac{-\pi}{T}, 0\right]$, $h(\omega, k)$ has the largest value when $k = n$. However, note that $h(\omega, 1) = h(-\omega, n)$. Thus, we can restrict the area of interest to $\omega \in \left[0, \frac{\pi}{T}\right]$. Let

$$f(\omega) := -h(\omega, 1) = 1 - \gamma^{1/n}\cos\left(\frac{\pi}{n} - \frac{\omega T}{n}\right) - \alpha(\gamma^{2/n} - (1 + \omega^2)). \tag{31}$$

Thus, if there exists an $\epsilon > 0$ such that $f(\omega) \geq \epsilon$ for all $\omega \in \left[0, \frac{\pi}{T}\right]$, then $H_\Pi \preceq -\epsilon I$ for all $\omega \in \mathbb{R}$.

For $\gamma > 1$, define

$$\overline{\omega} := \sqrt{\gamma^{2/n} - 1}$$
$$\overline{T} := \frac{\pi - n\arctan(\overline{\omega})}{\overline{\omega}}$$
$$\overline{\alpha} := \frac{\gamma^{1/n}\overline{T}}{2n\overline{\omega}}\sin\left(\frac{\pi - \overline{T}\overline{\omega}}{n}\right).$$

*Lemma 3:* If $1 < \gamma$, $\alpha = \overline{\alpha}$ and $T = \overline{T}$, then for $f$ in (31)

$$\operatorname*{argmin}_{\omega} f(\omega) = \overline{\omega}.$$

*Proof:* The lemma is proven true by inspecting the first, second and third derivatives of $f$ at $\overline{\omega}$. ∎

*Lemma 4:* If $\gamma > 1$, $\alpha = \overline{\alpha}$ and $T = \overline{T}$, then $\forall \omega \in \left[0, \frac{\pi}{T}\right]$

$$f(\omega) \geq 0.$$

*Proof:* Since $f(\overline{\omega}) = 0$ and $\overline{\omega}$ is the global minimum, $f(\overline{\omega}) \geq 0 \;\; \forall \overline{\omega} \in \left[0, \frac{\pi}{T}\right]$. ∎

*Lemma 5:* If $\gamma > 1$, for any $\hat{T}$ such that $0 \leq \hat{T} < \overline{T}$ ,

$$\overline{\alpha}\omega^2 - \gamma^{1/n}\cos\left(\frac{\omega\hat{T} - \pi}{n}\right) + 1 + \overline{\alpha} - \overline{\alpha}\gamma^{2/n} > 0 \tag{32}$$

for all $\omega \in \left[0, \frac{\pi}{T}\right]$.

*Proof:* The proof follows from inspecting (32) after substituting in for $y = \overline{T}\omega$. ∎

Therefore, if $T < \overline{T}$, $\alpha = \overline{\alpha}$ and

$$\epsilon := \min_{\omega} f(\omega), \tag{33}$$

then $f(\omega) \to \infty$ as $\omega \to \infty$ and, from Lemma 5, $f(\omega) > 0 \; \forall \omega \in \mathbb{R}$, by our choice of $T$ and $\alpha$. Hence, there exists an $\epsilon > 0$ and $\alpha_{i1} := 1, \alpha_{i2} := \alpha$ for $i = 1, \ldots, n$, such that $H_\Pi \preceq -\epsilon I$ holds for all $\omega \in \left[0, \frac{\pi}{T}\right]$. Furthermore, by the symmetry of $\lambda_k$ and Lemma 1, $H_\Pi \preceq -\epsilon I$ holds for all $\omega \in \mathbb{R}$.

## VII. Conclusions

We established a roll-off IQC, which describes a reduction in the gain with increasing frequency. The usefulness of the roll-off IQC was exhibited by Theorem 5 in an example where the interconnection structure was cyclic. As desired, when the roll-off IQC was combined with the OSP IQC, the resulting stability bound on the gain approached the secant condition for small delays and the small-gain condition for large delays.

In the future, we will investigate applications of the roll-off IQC to other network topologies. Additionally, we will investigate how to verify the roll-off IQC for a given state model. If the state model is described by polynomial vector functions, then elementary $\mathbf{L}_2$ gain methods [16], [15], combined with sum-of-squares (SOS) optimizations [14], [6], provide one method to verify an IQC.

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## IX. APPENDIX

We start with a lemma that will be used in Theorem 2.

*Lemma 6:* Let $G$ and $\tilde{G}$ represent two different cyclic matrices as in (13) such that

$$\prod_{i=1}^{n} g_i(j\omega) = \prod_{i=1}^{n} \tilde{g}_i(j\omega), \quad \tilde{g}_i(j\omega), g_i(j\omega) \neq 0. \tag{34}$$

There exists a diagonal nonsingular $D(j\omega) \in \mathbb{C}^{n\times n}$ such that

$$D(j\omega)^{-1}G(j\omega)D(j\omega) = \tilde{G}(j\omega). \tag{35}$$

*Proof:* Since (34) holds, choosing

$$d_1(j\omega) = 1, \tag{36}$$

$$d_i(j\omega) = d_{i-1}\frac{g_i(j\omega)}{\tilde{g}_i(j\omega)} \quad i = 2,\ldots,n \tag{37}$$

and $D(j\omega) = \operatorname{diag}(d_1(j\omega),\ldots,d_n(j\omega))$ provides a nonsingular, diagonal $D(j\omega)$ such that (35) holds. ∎

*Proof of Theorem 2:* ($\Rightarrow$) From Lemma 6, we know that there exists a nonsingular, diagonal $D(j\omega)$ such that (35) holds. Let $\overline{D}(j\omega) = I_2 \otimes D(j\omega)$. Since $D(j\omega)$ is nonsingular, multiplying (16) by $D(j\omega)$ will not effect the inequality constraint. Hence, the following holds for all $\omega$:

$$\begin{aligned} D^*(j\omega)\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* (\overline{D}^*(j\omega))^{-1}\overline{D}^*(j\omega)\Pi(j\omega)\overline{D}(j\omega) \\ \overline{D}(j\omega)^{-1}\begin{bmatrix} G(j\omega) \\ I \end{bmatrix} D(j\omega) \preceq -\epsilon D^*(j\omega)D(j\omega). \end{aligned} \tag{38}$$

The right hand term $-\epsilon D^*(j\omega)D(j\omega) = -\epsilon|d_i(j\omega)|^2 I$ is a constant, negative definite, diagonal matrix. Note from (14), (36), (37), that $|d_i(j\omega)|$ are constant scalars. Thus, we use the notation $|d_i|$. Let $\tilde{\epsilon} = \epsilon\max(|d_i|^2)$. Since $G(j\omega)$ and $\tilde{G}(j\omega)$ are similar for $D(j\omega)$, (38) implies

$$\begin{bmatrix} \tilde{G}(j\omega) \\ I \end{bmatrix}^* \overline{D}^*(j\omega)\Pi(j\omega)\overline{D}(j\omega)\begin{bmatrix} \tilde{G}(j\omega) \\ I \end{bmatrix} \preceq -\tilde{\epsilon}I \tag{39}$$

holds for all $\omega$. $D(j\omega)$ and $\alpha_{ik}\Pi_k(j\omega)$ are square. Therefore (39) holds if and only if

$$\begin{aligned} \begin{bmatrix} \tilde{G}(j\omega) \\ I \end{bmatrix}^* \sum_{i=1}^{n}\sum_{k=1}^{p} \alpha_{ik}\Pi_k(j\omega) \\ \otimes(D^*(j\omega)e_ie_i^TD(j\omega))\begin{bmatrix} \tilde{G}(j\omega) \\ I \end{bmatrix} \preceq -\tilde{\epsilon}I \end{aligned} \tag{40}$$

holds for all $\omega$. Since $D(j\omega)$ and $e_ie_i^T$ are are diagonal, $D^*(j\omega)e_ie_i^TD(j\omega) = |d_i(j\omega)|^2 e_ie_i^T$. Since, the $|d_i|$ terms are constant scalars, we move them to the front of the product. Hence, (40) holds if and only if

$$\begin{bmatrix} \tilde{G}(j\omega) \\ I \end{bmatrix}^* \sum_{i=1}^{n}\sum_{k=1}^{p} |d_i|^2\alpha_{ik}\Pi_k(j\omega)\otimes e_ie_i^T \begin{bmatrix} \tilde{G}(j\omega) \\ I \end{bmatrix} \preceq -\tilde{\epsilon}I$$

holds for all $\omega$. Therefore, $\tilde{\epsilon} = \epsilon\max(|d_i|^2)$ and $\tilde{\alpha}_{ik} := |d_i|^2\alpha_{ik}$ are the appropriate, constant positive and non-negative multiples for the condition in (17) to hold for all $\omega$. ($\Leftarrow$) From the symmetry, given the multipliers for (17), the multipliers of (16) can be recovered by the same argument. ∎