Further results on delay robustness of interconnected passive systems

Erin Summers, Murat Arcak, and Andrew Packard

***Abstract*— We analyze the stability of networks of passive systems with time delays in the interconnections. Building upon previous work where we complemented the passivity property with a "roll-off" integral quadratic constraint (IQC), here we expand our results to state space models. We first demonstrate how to verify that a state space model satisfies passivity and roll-off IQCs when the equilibrium is not known. We then illustrate the result on an Internet congestion control problem.**

I. INTRODUCTION

Integral quadratic constraints (IQCs) provide a method for characterizing input-output properties of a system. An advantage of the IQC stability test in [1] is that a composite IQC can be constructed to encapsulate all of the IQCs that a system satisfies, in order to obtain less conservative results than if the stability test were performed with each IQC individually. In [2], we proposed the joint use of an output strict passivity (OSP) and a roll-off IQC for the stability test of interconnected passive systems with time delays. By implementing the combined IQC, we achieved sharp stability estimates.

In this paper, we provide methods for verifying that a given state space system satisfies the output strict passivity IQC and the roll-off IQC. We focus on an equilibrium-independent verification the IQCs, since the equilibrium of the interconnection is sensitive to small perturbations in the subsystems and may not be accurately known. This is indeed a critical problem for biological networks where the parameters often exhibit wide variations, and for resource allocation algorithms in communication networks where the goal is to stabilize an optimal network equilibrium that is unknown to the users.

Establishing the stability of Internet congestion control algorithms has been a major research topic over the past decade [3], [4], [5], [6], [7], [8], [9]. A broadly applicable passivity approach was presented in [10], but this study did not take into account the forward and backward delays from the users to the routers. Stability estimates which bound the time delay are essential when achieving robustness and satisfactory performance for the Internet congestion control system, since time delay is an inherent property. As a motivating example, we apply both the IQC verification techniques and the IQC stability analysis to this problem.

Section III defines an interconnected system and reviews the IQC stability theorem. Section IV reviews the results from the previous paper [2]. Section V applies the results to state space models. Section VI outlines a method for verifying the output strict passivity and roll-off IQCs. An application to the Internet congestion control problem is provided in Section VII, and numerical results are presented in Section VIII. Finally, Section IX presents the conclusions.

E. Summers is with Department of Electrical Engineering, University of California, Berkeley, erin_summers@berkeley.edu

M. Arcak is with Department of Electrical Engineering, University of California, Berkeley, arcak@eecs.berkeley.edu

A. Packard is with Department of Mechanical Engineering, University of California, Berkeley, apackard@me.berkeley.edu

II. NOTATION AND DEFINITIONS

Let $\mathbb{N}$ be the set of natural numbers. Let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers. The set of $m \times n$ matrices whose elements are in $\mathbb{R}$ or $\mathbb{C}$ are denoted as $\mathbb{R}^{m\times n}$ and $\mathbb{C}^{m\times n}$. A single superscript index denotes vectors, e.g. $\mathbb{R}^m$ is the set of $m\times 1$ vectors whose elements are in $\mathbb{R}$. Let $e_i \in \mathbb{R}^n$ be the unit vector with zeros everywhere except in the *ith* element. Let $\mathbf{1}$ represent the vector whose entries are all one. For a matrix R let $[R]_{ij}$ denote the (i, j)th entry.

Basic system theory and functional analysis drawn from texts such as [11], [12] and [13] is used without further citation. $\mathbf{L}_2^m$ is the space of $\mathbb{R}^m$-valued functions $f : [0, \infty) \to \mathbb{R}^m$ of finite energy $\|f\|_2^2 = \int_0^\infty f(t)^T f(t)\, dt$. For $u \in \mathbf{L}_2^m$, $\hat{u}$ denotes the Fourier (Plancherel) transform of u. Associated with $\mathbf{L}_2^m$ is the *extended* space $\mathbf{L}_{2e}^m$, consisting of functions whose truncation $f_T(t) := f(t)$ for $t \le T$; $f_T(t) := 0$ for $t > T$, is in $\mathbf{L}_2^m$ for all $T > 0$.

For $u, v \in \mathbf{L}_2$, define $\langle u, v\rangle := \int_{-\infty}^{\infty} \hat{u}(\omega)^*\hat{v}(\omega)\ d\omega$, which is an inner product associated with the $\mathbf{L}_2$ norm. Let $\Pi : j\mathbb{R} \to \mathbb{C}^{(l+m)\times(l+m)}$ be a measurable, bounded Hermitian-valued function. A bounded, causal operator Δ mapping $\mathbf{L}_{2e}^l \to \mathbf{L}_{2e}^m$ is said to satisfy *the IQC defined by* Π, if for all $v \in \mathbf{L}_2^l$, with $y = \Delta v$, the inequality holds [1]:

$$\int_{-\infty}^{\infty} \begin{bmatrix}\widehat{v}(j\omega)\\ \widehat{y}(j\omega)\end{bmatrix}^* \Pi(j\omega) \begin{bmatrix}\widehat{v}(j\omega)\\ \widehat{y}(j\omega)\end{bmatrix} d\omega \ge 0.$$

III. INTERCONNECTED SYSTEMS

Consider Figure 1, where G has the form:

$$G(s) = \sum_{k=1}^{M} G_k(s)e^{-sT_k} + G_0(s), \tag{1}$$

where G_k for $k = 0, \ldots, M$ are proper, rational functions without poles in the closed right-half plane[1], and each Δ_i is a bounded, causal operator. Our main interest is in the situation where Δ_i are dynamical blocks representing the subsystems of a network, and G represents their interconnection structure, possibly containing delays.

[1]The authors in [1] consider a rational G, but the general form in (1) is admissible, as alluded to in [14].

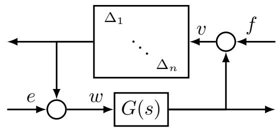

Fig. 1. The Δ_i blocks represent subsystems of a network and G represents the interconnection.

We now review the IQC stability test of [1], first recalling the definitions of well-posedness and stability of interconnected systems from [1].

Definition 1: The interconnection of G and Δ in Figure 1 is *well-posed* if the map $(v, w) \longmapsto (e, f)$ defined by

$$f = v - Gw, \quad (2)$$
$$e = w - \Delta v \quad (3)$$

has a causal inverse on $\mathbf{L}_{2e}$. The interconnection is *stable* if, in addition, the inverse is bounded.

From this point on, we assume that, for every $\tau \in [0, 1]$, the interconnection of G and $\tau\Delta$ is well-posed, as stipulated in [1]. The second condition in [1] is that, for every $\tau \in [0, 1]$, the IQC defined by Π is satisfied by $\tau\Delta$. The IQCs employed in this paper are structured such that if Δ satisfies the IQC, then so does $\tau\Delta$, $\tau \in [0, 1]$ (*cf.* [1, Remark 2]).

In particular, we focus on satisfying the third and final condition in [1]: if there exists an $\epsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \preceq \epsilon I, \quad (4)$$

then the interconnection of G and Δ is stable.

IV. Summary of Previous Results

In [2], we proposed the joint use of an output-strict passivity and a roll-off IQC in the stability test (4) for interconnections whose subsystems are OSP with a gain that rolls off. By analyzing the stability using the combined IQC, a less conservative bound on the time delay was achieved than analysis using either the OSP or the roll-off IQC alone. Combining the two IQCs is advantageous because both the time-scale information from the roll-off IQC and the phase information from the output-strict passivity IQC are captured at once. We outline the results of the previous paper [2] and build upon them throughout the paper.

Consider the following IQC:

$$\Pi_{1,\gamma_1} = \begin{bmatrix} 0 & 0.5 \\ 0.5 & -\frac{1}{\gamma_1} \end{bmatrix},$$

$\gamma_1 > 0$, which encapsulates an *output strict passivity* (OSP) property with $\mathbf{L}_2$ gain γ_1:

$$\langle y, v \rangle - \frac{1}{\gamma_1} \|y\|_2^2 \geq 0. \quad (5)$$

Likewise, we can encapsulate a roll-off property with "corner frequency" ω_c with the following IQC:

$$\Pi_{2,\gamma_2}(j\omega) := \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1+(\frac{\omega}{\omega_c})^2}{\gamma_2^2} \end{bmatrix},$$

which describes a reduction in the gain γ_2 with increasing frequency.

For systems that are both OSP and have roll-off, it is natural to use an IQC which combines Π_{1,γ_1} and Π_{2,γ_2}. Suppose each Δ_i satisfies the IQC defined by Π_{1,γ_1} and Π_{2,γ_2} with ω_c. Then the block diagonal concatenation $\Delta := \text{diag}(\Delta_i)$ satisfies the IQC defined by

$$\Pi(j\omega) := \sum_{i=1}^{n} (\alpha_{i1}\Pi_{1,\gamma_1} + \alpha_{i2}\Pi_{2,\gamma_2}(j\omega)) \otimes e_i e_i^T \quad (6)$$

for any choice of $\alpha_{i1} \geq 0, \alpha_{i2} \geq 0$, $i = 1, \ldots, n$.

To perform the stability test, we construct a frequency grid for a finite set of ω and search for $\alpha_{i1} \geq 0, \alpha_{i2} \geq 0$ such that (4) holds with the form in (6) for each ω in the grid. It may be numerically involved to employ a sufficiently dense grid for $\omega \in \mathbb{R}$. However, if the delays are commensurate[2], then it is sufficient to analyze (4) on a subset of $\mathbb{R}$, as detailed in [2].

V. Application to State Space Models

Several recent publications presented a passivity approach for overcoming the complexity of high-order differential equation models arising in communication networks [10], [16], [17], cooperative robotic vehicles [18], [19], [20], and biochemical reaction networks [21], [22]. This approach decomposes the network into passive components and applies a stability test that is equivalent to the IQC test of this paper with Π_{1,γ_1} only. We now generalize this method to systems with time delays and incorporate the roll-off IQC Π_{2,γ_2} in the stability analysis.

Let Δ_i refer to a dynamical system of the form:

$$\dot{x}_i = f_i(x_i, u_i)$$
$$y_i = h_i(x_i),$$

for $i = 1, \ldots, n$, where $x_i(t) \in \mathbb{R}^{n_i}$, $u_i(t) \in \mathbb{R}$, $y_i(t) \in \mathbb{R}$. Let $N = \sum_{i=1}^{n} n_i$ and define $u \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, $x \in \mathbb{R}^N$, by $u^T = [u_1 \ \ldots \ u_n]$, $y^T = [y_1 \ \ldots \ y_n]$, $x^T = [x_1^T \ \ldots \ x_n^T]$. The interconnection of these subsystems is described by the feedback law

$$u = G_0 y(t) + \sum_{k=1}^{M} G_k y(t - T_k),$$

where G_k are constant for $k = 0, \ldots, M$. Assume that the interconnected system possesses an equilibrium $x^\star$, and let $x_i^\star$ denote the part corresponding to the ith subsystem. Let:

$$y_i^\star := h_i(x_i^\star),$$
$$(y^\star)^T := [y_1^\star \ \ldots \ y_n^\star],$$
$$u^\star := \left(G_0 + \sum_{k=1}^{M} G_k \right) y^\star,$$
$$\bar{u}_i := u_i - u_i^\star,$$
$$\bar{y}_i := y_i - y_i^\star,$$
$$\bar{x}_i := x_i - x_i^\star.$$

[2]If all of the ratios between delays $\frac{T_i}{T_j}$ for $i, j = 1, \ldots, M$ are rational numbers, then the delays are said to be *commensurate* [15].

Let $\bar{\Delta}_i$ represent the Δ_i block with input $\bar{u}_i$ and output $\bar{y}_i$. In order to apply the IQC test discussed in Section IV, we need to verify the OSP and roll-off IQCs for each of the $\bar{\Delta}_i$ blocks, whose definitions depend on the network equilibrium $x^\star$ as shown above. However, the equilibrium of a network depends on the parameters of the subsystems. In many of our motivating applications, such as biological reaction networks and the Internet congestion control problem, the equilibrium of the network is not known *a priori* and it is essential to verify these IQCs without relying on the knowledge of $x^\star$. A procedure for equilibrium independent verification is presented next.

VI. Equilibrium Independent Verification of IQCs

Motivated by the discussion in Section V, we now study the equilibrium independent verification of Π_{1,γ_1} and Π_{2,γ_2}. In the previous sections we assumed that the Δ_i blocks are single-input single-output blocks for notational simplicity. However, this assumption is not essential, and identical results hold for m-input m-output blocks if G_k, $k = 0, \dots, M$ in (1) are replaced with $G_k \otimes I_m$ and the IQCs Π_i, $i = 1, 2$ are replaced with $\Pi_i \otimes I_m$. Thus, in this section we study m-input m-output blocks for further generality.

Let Δ refer to a dynamical system of the form:

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x), \end{aligned}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^m$. We assume that for all $u^\star \in \mathbb{R}^m$ there exists a unique $x^\star \in \mathbb{R}^n$ such that $f(x^\star, u^\star) = 0$.

We recall the following definition from [23]:

Definition 2: Δ is said to be *output strictly equilibrium independent passive* (OSEIP) with gain $\gamma_1 > 0$ if for every $u^\star \in \mathbb{R}^m$, there exists a once-differentiable storage function $S_{u^\star} : \mathbb{R}^n \to \mathbb{R}$ such that $S_{u^\star}(x) > 0 \; \forall x \neq x^\star$, $S_{u^\star}(x^\star) = 0$, and

$$\begin{aligned} \nabla_x S_{u^\star} \cdot f(x, u) \leq (u - u^\star)^T (y - y^\star) \\ - \frac{1}{\gamma_1} (y - y^\star)^T (y - y^\star) \quad (7) \end{aligned}$$

for all $u \in \mathbb{R}^m, x \in \mathbb{R}^n$, where $y^\star = h(x^\star)$. □

By integrating (7) with respect to time from 0 to ∞, and assuming $\bar{x}(0) = 0$, it is not difficult to verify that (5) holds for the input-output pair $\bar{u}, \bar{y}$, which means that $\bar{\Delta}$ satisfies the IQC defined by $\Pi_{1,\gamma_1} \otimes I_m$. Clearly, the assumption $\bar{x}(0) = 0$ must be eliminated for stability analysis of the interconnected system in Figure 1. This can be done by assuming an appropriate *reachability* property for the interconnected system, which is a standard approach in going from input-output to state space stability [24, Section 6.3], [25]. In this approach one treats the system as if it had zero initial conditions and uses reachability to prescribe bounded and finite-duration (therefore, $\mathbf{L}_2$) exogenous signals that bring the state to the actual, non-zero initial condition. One then uses the input-output stability of the interconnection to conclude that the internal signals are in $\mathbf{L}_2$, and employs additional reachability or detectability conditions for the subsystems to conclude that the states converge to zero.

In order to verify that $\bar{\Delta}$ satisfies Π_{2,γ_2}, we cascade $\bar{\Delta}$ with a linear system whose gain "rolls-up" with corner frequency ω_c, illustrated in Figure 2. Next, we estimate the gain from the input $\bar{u}$ to the output of the roll-up system $\bar{z}$. A bounded $\mathbf{L}_2$ gain from $\bar{u}$ to $\bar{z}$ implies that $\bar{\Delta}$ satisfies Π_{2,γ_2} with corner frequency ω_c.

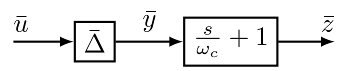

Fig. 2. Cascade of Δ with roll-up. If the gain from $\bar{u}$ to $\bar{z}$ is bounded, then $\bar{\Delta}$ rolls off at ω_c.

The state equations for the new cascaded system are

$$\dot{\bar{x}} = F_{x^* u^*}(\bar{x}, \bar{u}) := f(x, u) - f(x^\star, u^\star) \quad (8)$$

$$\bar{z} := \frac{\dot{\bar{y}}}{\omega_c} + \bar{y} \quad (9)$$

To verify the roll-off IQC, we bound the $\mathbf{L}_2$ gain of the cascaded system dynamics (8)-(9) from $\bar{u}$ to $\bar{z}$.

Definition 3: Δ is said to have *equilibrium independent roll-off* with corner frequency $\omega_c > 0$ and gain $\gamma_2 > 0$ if for every $u^\star \in \mathbb{R}^m$, there exists a once-differentiable storage function $V_{u^\star}(x) : \mathbb{R}^n \to \mathbb{R}$ such that $V_{u^\star}(x) > 0 \; \forall x \neq x^\star$, $V_{u^\star}(x^\star) = 0$, and

$$\begin{aligned} 0 \leq -\nabla V_{u^\star} F_{x^*, u^*}(\bar{x}, \bar{u}) + \gamma_2^2 (u - u^\star)^T (u - u^\star) \\ - (z - z^\star)^T (z - z^\star) \quad (10) \end{aligned}$$

$\forall x \in \mathbb{R}^n, \forall u \in \mathbb{R}^m$, where $z^\star = \frac{\dot{y}^\star}{\omega_c} + y^\star$. We notate this as EIRO(γ_2, ω_c).

Thus, the EIRO(γ_2, ω_c) property may be used to show that $\bar{\Delta}$ satisfies $\Pi_{2,\gamma_2} \otimes I_m$ with corner frequency ω_c regardless of where the equilibrium is located.

As a special case of practical interest, we focus on the nonlinear system Δ with dynamics

$$\dot{x} = -\beta x - \phi(x) + u \quad (11)$$

$$y = x, \quad (12)$$

where $x \in \mathbb{R}$, $u \in \mathbb{R}$, $y \in \mathbb{R}$, $\beta > 0$, $\phi(x) : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing so that for all $(x, x^\star) \in \mathbb{R} \times \mathbb{R}$,

$$(x - x^\star)(\phi(x) - \phi(x^\star)) \geq 0. \quad (13)$$

Claim 1: $\bar{\Delta}$ satisfies $\Pi_{1,\frac{1}{\beta}}$.

Proof: Consider the storage function

$$S_{u^\star}(x) := \frac{1}{2}(x - x^\star)^2. \quad (14)$$

Clearly $S_{u^\star}(x) > 0$ for all $x \neq x^\star$ and $S_{u^\star}(x^\star) = 0$. We show that the condition in (7) holds for all $x \in \mathbb{R}, u \in \mathbb{R}$ for the system Δ using the storage function (14).

We subtract $f(x^\star, u^\star)$, which is equal to zero, from (11) and multiply by $\nabla S_{u^\star}(x) = \bar{x}$, which yields

$$\nabla S_{u^\star}(x) \left(f(x, u) - f(x^\star, u^\star) \right) \leq -\beta \bar{x}^2 + \bar{x}\bar{u}$$

from (13). Thus, Δ is OSEIP with $\gamma_1 = \frac{1}{\beta}$. ■

Claim 2: $\bar{\Delta}$ satisfies $\Pi_{2,\frac{1}{\beta}}$ with $\omega_c > \beta$.

Proof: After cascading $\bar{\Delta}$ with the roll-up with ω_c, the new dynamics are

$$\dot{\bar{x}} = -\beta\bar{x} - (\phi(x) - \phi(x^\star)) + \bar{u} \quad (15)$$

$$\bar{z} = \left(1 - \frac{\beta}{\omega_c}\right)\bar{x} - \frac{\phi(x) - \phi(x^\star)}{\omega_c} + \frac{\bar{u}}{\omega_c}. \quad (16)$$

With the choice of storage function

$$V_{u^\star}(x) = 2\gamma_2^2 \int_{x^\star}^{x} [\phi(y) - \phi(x^\star)]dy + \left(\beta\gamma_2^2 - \frac{1}{\omega_c}\right)\bar{x}^2,$$

we show that $V_{u^\star}(x^\star) = 0$, $V_{u^\star}(x) > 0$ for all $x \neq x^\star$, and (10) hold.

Trivial inspection of $V_{u^\star}(x)$ reveals that $V_{u^\star}(x^\star) = 0$ and $V_{u^\star}(x) > 0$ for all $x \neq x^\star$.

Let $\bar{\phi}_{x^\star}(x) := \phi(x) - \phi(x^\star)$. Note that right hand side of (10) is a quadratic function in $\bar{u}$ and the inequality holds for all u. Thus, we can apply the Bounded Real Lemma and eliminate the dependence on $\bar{u}$. Assume $\gamma_2 = \frac{1}{\beta}$ and $\gamma_2\omega_c > 1$. Then (10) holds for all $x \in \mathbb{R}$ and $u \in \mathbb{R}$ if and only if both

$$\gamma_2^2\omega_c^2 > 1, \text{ and} \quad (17)$$

$$\left[\frac{2}{\omega_c}\left(\bar{x} - \frac{\beta\bar{x} + \bar{\phi}_{x^\star}(x)}{\omega_c}\right) + \nabla V_{u^\star}(x)\right]^2 + 4\left[\gamma_2^2 - \frac{1}{\omega_c^2}\right]\left(\bar{x} - \frac{\beta\bar{x} + \bar{\phi}_{x^\star}(x)}{\omega_c}\right)^2 + 4\left[\gamma_2^2 - \frac{1}{\omega_c^2}\right]\nabla V_{u^\star}(x)(-\beta\bar{x} - \bar{\phi}_{x^\star}(x)) \leq 0 \quad (18)$$

hold. The conditions (17) and (18) ensure that right hand side of (10) is convex and has roots that are imaginary or zero. Thus, the right hand side of (10) is greater than or equal to zero for all $x \in \mathbb{R}, u \in \mathbb{R}$. By assumption, (17) holds. Note that

$$\nabla V_{u^\star}(x) = 2\gamma_2^2\bar{\phi}_{x^\star}(x) + 2\left(\beta\gamma_2^2 - \frac{1}{\omega_c}\right)\bar{x}.$$

Substituting $\nabla V_{u^\star}(x)$ into (18) and expanding yields an inequality of the form

$$c_1\bar{x}^2 + c_2\bar{x}\bar{\phi}_{x^\star}(x) + c_3\bar{\phi}_{x^\star}^2(x) \leq 0, \quad (19)$$

where

$$c_1 = \frac{4}{\omega_c^2}\left(1 - \gamma_2^2\omega_c^2\right)(\beta^2\gamma_2^2 - 1),$$

$$c_2 = \frac{8\beta\gamma_2^2}{\omega_c^2}\left(1 - \gamma_2^2\omega_c^2\right),$$

$$c_3 = \frac{4\gamma_2^2}{\omega_c^2}\left(1 - \gamma_2^2\omega_c^2\right).$$

Since $\gamma_2\omega_c > 1$ and $\gamma_2^2\beta^2 = 1$, the terms c_1, c_2, and c_3 are less than or equal to zero. From the condition in (13), $\bar{x}\bar{\phi}_{x^\star}(x) \geq 0$. Hence, (19) holds for all $\bar{x}$ and (18) holds for all x.

Since both (17) and (18) hold, (10) holds for all $x \in \mathbb{R}$ and $u \in \mathbb{R}$. Hence, and $\bar{\Delta}$ satisfies $\Pi_{2,\frac{1}{\beta}}$ with $\omega_c > \beta$. ■

Note that the special case $\phi(x) \equiv 0$, in (11)-(12) gives a linear system with the transfer function $\frac{1}{s+\beta}$. In this case, the conditions $\gamma_2\omega_c > 1$, and $\gamma_2\beta > 1$ imply

$$\left|\frac{1}{s+\beta}\right|_{s=j\omega} < \left|\frac{\gamma_2}{\frac{s}{\omega_c}+1}\right|_{s=j\omega} \quad \forall\omega, \quad (20)$$

which is a roll-off with gain γ_2 and corner frequency ω_c.

VII. Application to Internet Congestion Control

Consider a set of N_U users and N_L links. The Internet congestion control problem is to design update algorithms for the sending rates x_i for $i = 1, \ldots, N_U$, and prices p_l for $l = 1, \ldots, N_L$, that are decentralized and have a stable network equilibrium that maximizes the aggregate utility

$$\sum_{i=1}^{N_U} U_i(x_i),$$

subject to the capacity constraints of the links, where $U_i(\cdot)$ is a concave utility function for user i.

The routing matrices $R_f \in \mathbb{R}^{N_L \times N_U}$ and $R_b \in \mathbb{R}^{N_L \times N_U}$ indicate if user i uses link l, and if so, how much forward delay T_{li}^f and backward delay T_{li}^b are contained are the link as follows:

$$[R_f(s)]_{li} = \begin{cases} e^{-T_{li}^f s} & \text{if user } i \text{ uses link } l \\ 0 & \text{else.} \end{cases} \quad (21)$$

$$[R_b(s)]_{li} = \begin{cases} e^{-T_{li}^b s} & \text{if user } i \text{ uses link } l \\ 0 & \text{else.} \end{cases} \quad (22)$$

Hence, the interconnection of the sources and links is described be the feedback law

$$\begin{bmatrix} q \\ z \end{bmatrix} = \begin{bmatrix} 0 & R_b^T \\ R_f & 0 \end{bmatrix}\begin{bmatrix} x \\ p \end{bmatrix},$$

where $q := [q_1^T \ldots q_{N_U}^T]^T$, $x := [x_1^T \ldots x_{N_U}^T]^T$, $z := [z_1^T \ldots z_{N_L}^T]^T$, and $p := [p_1^T \ldots p_{N_L}^T]^T$. Here q_i denotes the price feedback received by user i, and z_l is the aggregate rate for link l.

We now verify Π_{1,γ_1} and Π_{2,γ_2} for a class of user algorithms:

$$\dot{x}_i = U_i'(x_i) - q_i \quad (23)$$

$$y_i = x_i. \quad (24)$$

Let Ω_i refer to the dynamics in (23)-(24) with input $-q_i$. We assume $U_i''(x_i) \leq -\beta_i$ for all $x_i \in \mathbb{R}$ and for some $\beta_i > 0$, which implies that $U_i(\cdot)$ is concave. We further assume that $U_i'(x_i) \to \infty$ as $x_i \to 0^+$, which renders $\mathbb{R}_+^{N_U}$ invariant and allows us to restrict our analysis to this domain.

Claim 3: $\bar{\Omega}_i$ satisfies $\Pi_{1,\frac{1}{\beta}}$ and $\Pi_{2,\frac{1}{\beta}}$ with $\omega_c > \beta$.

Proof: Since $U_i(\cdot)$ is concave, $U_i'(\cdot)$ can be written as:

$$U_i'(x_i) = -\beta_i x_i - \phi_i(x_i),$$

where ϕ_i is a nondecreasing function because the derivative of ϕ_i is

$$\phi_i'(x_i) \quad = \quad -U_i''(x_i) - \beta_i \geq 0.$$

Hence, $\phi_i(x_i)$ is strictly increasing and (13) holds. Clearly, Ω_i is a special case of the system in (11)-(13). Thus, by Claim 1 and Claim 2, $\bar{\Omega}_i$ satisfies $\Pi_{1,\frac{1}{\beta}}$ and $\Pi_{2,\frac{1}{\beta}}$ with $\omega_c > \beta$. ■

Several classes of congestion control algorithms treat the links as static operators, interpreted as penalty functions that keep the link rate below capacity, and the users as dynamic operators [4]. For the link price we select the algorithm:

$$p_l \quad = \quad h_l(z_l), \tag{25}$$

where $h_l(\cdot)$ is a monotone penalty function. Let Δ_l describe the dynamics of the link price with input z_l and output p_l. Clearly, a static system will not satisfy the roll-off IQC Π_{2,γ_2}. However, since $h_l(\cdot)$ is strictly increasing, if the slope of h_l is less than or equal to γ_1, then $\bar{\Delta}_l$ satisfies Π_{1,γ_1}.

Let

$$\Sigma \quad := \quad \text{diag}(\Omega_1, \ldots, \Omega_{N_U}, \Delta_1, \ldots, \Delta_{N_L}),$$

and note that the interconnection matrix is

$$G \quad := \quad \begin{bmatrix} 0 & -R_b^T \\ R_f & 0 \end{bmatrix}, \tag{26}$$

since the input to Ω_i is $-q_i$.

To test the stability of the interconnection of Σ with G, we let

$$\Pi(j\omega) := \sum_{i=1}^{N_U} (\alpha_{i1}\Pi_{1,\frac{1}{\beta}} + \alpha_{i2}\Pi_{2,\frac{1}{\beta}}(j\omega)) \otimes e_i e_i^T + \sum_{i=N_U+1}^{N_U+N_L} (\alpha_{i1}\Pi_{1,\gamma_1}(j\omega)) \otimes e_i e_i^T,$$

where $\Pi(j\omega)$ is the composite IQC for the entire system of N_U users using N_L links. We then search for $\alpha_{i1} \geq 0$ for $i = 1, \ldots N_U + N_L$ and $\alpha_{j2} \geq$ for $j = 1, \ldots, N_U$ such that (4) holds for all $\omega \in \mathbb{R}$.

VIII. Numerical Example

As an example, we consider the interconnection in Figure 3 with one link serving N_U users. We combine the forward and backward delays to yield the round trip time

$$T_i \quad = \quad T_i^f + T_i^b. \tag{27}$$

In this special case, we can equivalently test the stability for an interconnection where all of the delay is in the forward routing matrix R_f.

Lemma 1: Let $G(j\omega)$ and $\tilde{G}(j\omega)$ represent two different interconnections as in (26) where $R_f, R_b, \tilde{R}_f, \tilde{R}_b$ are $r \times 1$ and

$$[R_f(s)]_i[R_b(s)]_i = [\tilde{R}_f(s)]_i[\tilde{R}_b(s)]_i \quad \forall i = 1, \ldots, r. \tag{28}$$

Let $\{\Pi_k(j\omega)\}_{k=1}^p$ represent an arbitrary set of IQCs.

Define

$$\Pi(j\omega) = \sum_{i=1}^{r+1}\sum_{k=1}^{p} \alpha_{ik}\Pi_k(j\omega) \otimes e_i e_i^T \tag{29}$$

and

$$\tilde{\Pi}(j\omega) = \sum_{i=1}^{r+1}\sum_{k=1}^{p} \tilde{\alpha}_{ik}\Pi_k(j\omega) \otimes e_i e_i^T. \tag{30}$$

There exist constants $\alpha_{ik} \geq 0$ and $\epsilon > 0$ such that for all $\omega \in \mathbb{R}$

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \preceq -\epsilon I, \tag{31}$$

if and only there if exist constants $\tilde{\alpha}_{ik} \geq 0$ and $\tilde{\epsilon} > 0$ such that for all $\omega \in \mathbb{R}$

$$\begin{bmatrix} \tilde{G}(j\omega) \\ I \end{bmatrix}^* \tilde{\Pi}(j\omega) \begin{bmatrix} \tilde{G}(j\omega) \\ I \end{bmatrix} \preceq -\tilde{\epsilon} I. \tag{32}$$

□

Proof: The proof is omitted for brevity. ■

We further assume that $T_i := T$ for $i = 1, \ldots, N_U$. By Lemma 1, we combine all of the forward and backward delay into R_f such that $R_f = e^{-sT}\mathbf{1^T}$ and $R_b = \mathbf{1}$ and test the stability of the new interconnection.

The user algorithms Ω_i are defined by (23) for $i = 1, \ldots, N_U$ with $U_i''(x_i) \leq -\beta$ for some $\beta > 0$ and for all $x_i \in \mathbb{R}$. By Claim 3, $\bar{\Omega}_i$ satisfies $\Pi_{1,\frac{1}{\beta}}$ and $\Pi_{2,\frac{1}{\beta}}$ with $\omega_c > \beta$ for $i = 1, \ldots, N_U$. The link price algorithm Δ_1 is defined by (25), where $h_l(\cdot)$ is a monotone penalty function with a slope less than or equal to γ_1. Thus, Δ_1 satisfies Π_{1,γ_1}. Since there is only one link and the link only satisfies one IQC, we incorporate the IQC gain γ_1 in the feedback loop in Figure 3 and assume henceforth that Δ_1 satisfies $\Pi_{1,1}$.

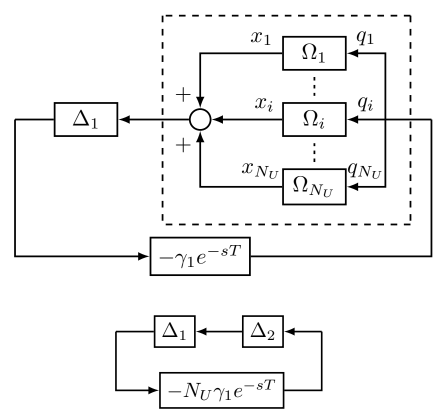

Fig. 3. Simplified network with one link and N_U users with round trip time T for each user. Stability is tested on the two system interconnection.

The stability test for this interconnection can be equivalently expressed as the stability test for an interconnection of two subsystems.

Lemma 2: Suppose there exist bounded, causal operators, $\{\Omega_i\}_{i=1}^K$ that satisfy the IQCs defined by Π_{1,γ_1} and Π_{2,γ_2} with ω_c. Then, for all $\theta_i \geq 0$, such that $\sum_{i=1}^K \theta_i = 1$, the bounded, causal operator $\Delta := \sum_{i=1}^K \theta_i\Omega_i$ also satisfies the IQCs defined by Π_{1,γ_1} and Π_{2,γ_2} with ω_c.

Proof: The proof is omitted for brevity. ■

By Lemma 2, if Ω_i for $i = 1, \dots, N_U$ in Figure 3 satisfy the IQCs defined by Π_{1,γ_1} and Π_{2,γ_2} with ω_c, then we can replace the sum of the users Ω_i in the dashed box by the subsystem $N_U\Delta_2$, where

$$\Delta_2 = \sum_{i=1}^{N_U} \frac{1}{N_U}\Omega_i, \tag{33}$$

and Δ_2 satisfies Π_{1,γ_1} and Π_{2,γ_2} with ω_c. This yields an interconnection of Δ_1 and Δ_2 in negative feedback with gain $N_U\gamma_1$ and delay T, seen in Figure 3.

As a numerical example, we performed the robustness test in (4) on the interconnection in Figure 3 for $N_U = 10$, with Δ_1 satisfying $\Pi_{1,1}$ and Δ_2 satisfying $\Pi_{1,\frac{1}{\beta}}$ and $\Pi_{2,\frac{1}{\beta}}$ at $\omega_c = 1$. We gridded the frequency $\omega \in \left[-\frac{\pi}{T}, \frac{\pi}{T}\right]$ with 300 points and varied $T \in (10^{-3}, 10^{3})$, which is illustrated in Figure 4. Since the delays are commensurate, we evaluate (4) over this subset of $\mathbb{R}$. If the number of sources N_U increases, the stability bound will decrease.

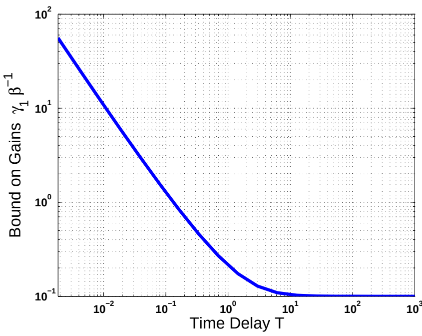

Fig. 4. Robustness test for the Internet congestion control problem with a static link (25) and dynamic user algorithm in (23) with $N_U = 10$. The bound on the gain $\gamma_1\beta^{-1} \to \frac{1}{N_U}$ as the time delay $T \to \infty$.

The global nonlinear results on the stability of the Internet congestion control problem in [7], [9], [8] are not comparable to this study, as they assumed a dynamic link price. Under similar dynamic user and static link algorithms assumptions, [6] analyzed the global asymptotic stability of the Internet congestion control problem with delays by using a small gain technique. However, in [6] the second derivative of the utility function is bounded below, which was not a restriction in our example. Thus, we achieve a less conservative stability estimate by removing this restriction.

IX. Conclusions

We presented an equilibrium independent verification of the OSP and roll-off IQCs, given a state space model of the subsystems. This is important because in many of our motivating applications, such as biological networks and Internet congestion control, the equilibrium of an interconnection is not known *a priori*. Equilibrium independent verification of the output strict passivity IQC can be achieved through methods outlined in [23]. The roll-off IQC was verified here by cascading the system of interest with a system whose output "rolls-up", and by estimating the bound on the gain for the cascaded system. We presented a motivating example of Internet congestion control, and discussed how the different IQC gains affect the overall delay.

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