On Time Delay Margin Estimation for Adaptive Control and Robust Modification Adaptive Laws

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Abstract

This paper presents methods for estimating time delay margin for adaptive control of input delay systems with almost linear structured uncertainty and a step input reference command signal. The bounded linear stability analysis method seeks to represent an adaptive law by a locally bounded linear approximation within a time window. The time delay margin of this input delay system represents a local stability measure and is computed analytically by three methods: Padé approximation, Lyapunov-Krasovskii method with sum-of-squares optimization, and matrix measure method. These methods are applied to the standard model-reference adaptive control, σ-modification adaptive law, and optimal control modification adaptive law. The windowing analysis results in non-unique estimates of the time delay margin since it is dependent on the length of a time window and parameters which vary from one time window to the next. The optimal control modification adaptive law overcomes this limitation in that, as the adaptive gain tends to infinity and if the matched uncertainty is linear, then the closed-loop input delay system tends to a linear time-invariant system. A lower bound of the time delay margin of this system can then be estimated uniquely without the need for the windowing analysis. Simulation results demonstrates the feasibility of the bounded linear stability method for time delay margin estimation.

1 Introduction

Input delay systems are generally non-minimum phase. For linear input delay systems, feedback gain must be kept to a reasonable value to maintain stability. Input delay influences stability of adaptive control in a similar manner. Adaptive gain is used to control the rate of adaptation in adaptive control. For model-reference adaptive control, it is well-known that as the adaptive gain increases, the closed-loop system loses robustness, thereby rendering it susceptible to instability in the presence of unmodeled dynamics and or input time delay. Thus, to maintain stability of an input delay adaptive system, the adaptive gain must be carefully selected. For a given value of the adaptive gain, there exists a corresponding value of input time delay for which the adaptive system is on the verge of instability. This is known as a time delay margin. To maintain stability, the adaptive gain of the system must be kept below the value that corresponds to the time delay margin of the system.

Global stability analysis for input delay adaptive systems is a challenging problem. Lyapunov-Krasovskii method or Lyapunov-Razumikhin method are much more difficult to apply to an adaptive system. Even for a simple scalar linear time-invariant (LTI) system, both the Lyapunov-Krasovskii method or Lyapunov-Razumikhin method can result in conservative estimates of the time delay margin [1]. So, even if a global stability analysis for input delay adaptive control is available, the conservatism in the estimation could render it impractical. The lack of available analytical methods for computing the time delay margin of adaptive control is a hurdle for certification of adaptive control [2].

While global stability analysis is challenging, several studies have recently been done to address local stability of input delay adaptive systems. One such method applies a Padé approximation to transform an input delay system into a delay-free higher-order system [3]. The transformed system is then analyzed using the standard Lyapunov method to estimate the time delay margin. However, this approach yields a highly conservative time delay margin even for

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a simple scalar adaptive system [3] thereby rendering the approach rather less practical. Another method, called bounded linear stability analysis, attempts to analyze the stability margins of an adaptive system in a local context [4, 5]. The method approximates an adaptive system as a series of bounded linear systems inside time windows. The windowing analysis allows the bounded linear systems to be analyzed using linear analytical tools. The method has been shown to provide a less conservative estimate of the time delay margin. Subsequently, a similar method has been developed using the windowing approach to estimate the stability margin of an adaptive system [6]. Linear matrix inequality (LMI) methods also have been used to analyze stability of adaptive control [7]. The Lyapunov-Razumikhin method has been used to estimate the time delay margin for a simple scalar adaptive system [8]. The method requires optimization of the candidate Lyapunov function in order to reduce the conservatism in the estimated time delay margin.

This paper extends the bounded linear stability analysis method for analyzing input delay adaptive control with almost linear structured uncertainty. Stability of an input delay adaptive system with a step input reference command signal is analyzed by three methods: Padé approximation, Lyapunov-Krasovskii method with sum-of-squares optimization, and the matrix measure method, to estimate the local time delay margin of a bounded linear system inside each time window. Three different adaptive laws are used: the standard MRAC, σ-modification adaptive law [9], and optimal control modification adaptive law [10]. Asymptotic analysis of the time delay margin as the adaptive gain tends to infinity is performed to study to effect of large adaptive gain on the time delay margin. Simulations are studied to demonstrate the feasibility of the time delay margin estimation.

2 Input Delay Adaptive Systems and Bounded Linearity Stability Analysis

Given an input delay nonlinear plant

$$\dot{x}(t) = Ax(t) + B\left[u(t - t_d) + \Theta^{*\top}\Phi(x(t))\right] \tag{1}$$

where $x(t) : [0,\infty) \to \mathbb{R}^n$ is a state vector, $u(t) : [0,\infty) \to \mathbb{R}^p$ is a control vector, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times p}$ are known such that the pair (A,B) is controllable, $\Theta^* \in \mathbb{R}^{m\times p}$ is an unknown constant weight matrix that represents a parametric uncertainty, $\Phi(x(t)) : \mathbb{R}^n \to \mathbb{R}^m$ is a vector of known functions, and t_d is an input time delay.

The structure of the uncertainty is assumed to be linearly dominant. That is

$$\Phi(x(t)) = x(t) + \delta(x) \tag{2}$$

where $\|\delta(x)\| \ll \|x(t)\|$ is small.

The input delay t_d could also be viewed as the time delay margin for robustness against unmodeled dynamics of the delay-free system

$$\dot{x}(t) = Ax(t) + B\left[u(t) + \Theta^{*\top}\Phi(x(t))\right] \tag{3}$$

The reference model is specified as

$$\dot{x}_m(t) = A_m x_m(t) + B_m r(t) \tag{4}$$

where $A_m \in \mathbb{R}^{n\times n}$ is Hurwitz and known, $B_m \in \mathbb{R}^{n\times p}$ is also known, and $r(t) : [0,\infty) \to \mathbb{R}^p \in \mathscr{L}_\infty$ is a bounded command vector.

Defining the tracking error as $e(t) = x_m(t) - x(t)$, then the controller $u(t)$ is specified by

$$u(t) = -K_x x(t) + K_r r(t) - u_{ad}(t) \tag{5}$$

where $K_x \in \mathbb{R}^{p\times n}$ and $K_r \in \mathbb{R}^{p\times p}$ are known nominal gain matrices, and $u_{ad}(t) \in \mathbb{R}^p$ is an adaptive signal given by

$$u_{ad}(t) = \Theta^\top(t)\Phi(x) \tag{6}$$

where $\Theta \in \mathbb{R}^{m\times p}$ is an estimate of Θ^*.

Assuming that the model matching conditions can be satisfied, then

$$A_m = A - BK_x \tag{7}$$

$$B_m = BK_r \tag{8}$$

The standard model-reference adaptive control law is

$$\dot{\Theta}(t) = -\Gamma\Phi(x(t))e^{\top}(t)PB \tag{9}$$

where $\Gamma = \Gamma^{\top} \in \mathbb{R}^{m\times m} > 0$ and $P = P^{\top} \in \mathbb{R}^{n\times n} > 0$ solves the Lyapunov equation

$$PA_m + A_m^{\top}P = -Q \tag{10}$$

where $Q = Q^{\top} \in \mathbb{R}^{n\times n} > 0$.

The error equation corresponding to the input delay system (3) can be derived by substituting the time-delay version of the controller from Eq. (5), thus resulting in

$$\dot{e}(t) = Ae(t) - BK_x e(t-t_d) + Bu_{ad}(t-t_d) - B\Theta^{*\top}\Phi(x(t)) - BK_x[x_m(t) - x_m(t-t_d)] + BK_r[r(t) - r(t-t_d)] \tag{11}$$

To analyze this system, the bounded linear stability analysis method has been proposed to approximate the adaptive system as a series of bounded linear systems within time windows [4, 5]. The windowing analysis then permits the use of linear tools to analyze stability of the approximated bounded linear systems inside the time windows.

Theorem 1: The adaptive law (9) is bounded locally by a linear approximation as

$$\dot{\Theta}^{\top}(t)\Phi(x(t)) = -\gamma B^{\top}Pe(t) \tag{12}$$

where γ is a constant defined locally and retrospectively as

$$\gamma = \frac{1}{T_0}\int_{t_i-T_0}^{t_i}\Phi^{\top}(x(\tau))\Gamma\Phi(x(\tau))\,d\tau \tag{13}$$

for $t \in [t_i - T_0, t_i)$, where $t_0 = 0$, $t_i = t_{i-1} + T_0$ and $i = 1, 2, \ldots, n \to \infty$.

Proof: Choose a Lyapunov candidate function

$$V(t) = e^{\top}(t)Pe(t) + \text{trace}\left[\tilde{\Theta}^{\top}(t)\Gamma^{-1}\tilde{\Theta}(t)\right] \tag{14}$$

The error equation of the delay-free system is

$$\dot{e}(t) = A_m e(t) + B\tilde{\Theta}^{\top}(t)\Phi(x(t)) \tag{15}$$

where $\tilde{\Theta}(t) = \Theta(t) - \Theta^{*}$.

Denoting $V_g(t)$ as the Lyapunov candidate function to be evaluated globally using the adaptive law (9) as follows:

$$\dot{V}_g(t) = -e^{\top}(t)Qe(t) + 2e^{\top}(t)PB\tilde{\Theta}^{\top}(t)\Phi(x(t)) - 2\text{trace}\left[\tilde{\Theta}^{\top}(t)\Phi(x(t))e^{\top}(t)PB\right] = -e^{\top}(t)Qe(t) \le 0 \tag{16}$$

Denoting $V_l(t)$ as the Lyapunov candidate function to be evaluated locally with a time window using the locally bounded linear approximation (12) yields

$$\begin{aligned}\dot{V}_l(t) &= -e^{\top}(t)Pe(t) + 2e^{\top}(t)PB\tilde{\Theta}^{\top}(t)\Phi(x(t)) - 2\text{trace}\left[\tilde{\Theta}^{\top}(t)\Gamma^{-1}\dot{\tilde{\Theta}}(t)\right] \\ &= \dot{V}_g(t) + 2e^{\top}(t)PB\tilde{\Theta}^{\top}(t)\Phi(x(t)) - 2\text{trace}\left\{\tilde{\Theta}^{\top}(t)\Phi(x(t))\left[\Phi^{\top}(x(t))\Gamma\Phi(x(t))\right]^{-1}\Phi^{\top}(x(t))\dot{\tilde{\Theta}}(t)\right\} \\ &= \dot{V}_g(t) + 2e^{\top}(t)PB\tilde{\Theta}^{\top}(t)\Phi(x(t))\left\{1 - \gamma\left[\Phi^{\top}(x(t))\Gamma\Phi(x(t))\right]^{-1}\right\}\end{aligned} \tag{17}$$

for $t \in [t_i - T_0, t_i)$, where $t_0 = 0$, $t_i = t_{i-1} + T_0$ and $i = 1, 2, \ldots, n \to \infty$.

Consider the integral form of Eq. (17)

$$\int_{t_i-T_0}^{t_i}\dot{V}_l(t)\,dt = \int_{t_i-T_0}^{t_i}\dot{V}_g(t)\,dt + \int_{t_i-T_0}^{t_i}2e^{\top}(t)PB\tilde{\Theta}^{\top}(t)\Phi(x(t))\left\{1 - \gamma\left[\Phi^{\top}(x(t))\Gamma\Phi(x(t))\right]^{-1}\right\}dt \tag{18}$$

Then for $V_g(t)$ and $V_l(t)$ to be equal in a time window

$$\int_{t_i-T_0}^{t_i}\dot{V}_l(t)\,dt = \int_{t_i-T_0}^{t_i}\dot{V}_g(t)\,dt \tag{19}$$

which implies

$$\int_{t_i-T_0}^{t_i} 2e^\top(t) PB\tilde{\Theta}^\top(t)\Phi(x(t))\left\{1-\gamma\left[\Phi^\top(x(t))\Gamma\Phi(x(t))\right]^{-1}\right\}dt=0 \tag{20}$$

It is important to note that this is a definite integral equation for which a valid solution can include a constant solution of γ. Such a solution is called a "weak-form" or integral-form solution which is valid only over a finite time interval. In contrast, the "strong-form" solution is a global solution that satisfies for all time. In the windowing analysis, the weak-form solution is used.

The mean value theorem for integration states that

$$\int_a^b F(t)G(t)dt = F(c)\int_a^b G(t)dt \tag{21}$$

where $c\in[a,b]$ and $\mathrm{G}(t)\geq 0$.

Let $\bar{t}\in[t_i-T_0,t_i)$, then applying the mean value theorem for integration to Eq. (20) yields

$$\int_{t_i-T_0}^{t_i} 2e^\top(t)PB\tilde{\Theta}^\top(t)\Phi(x(t))\left[\Phi^\top(x(t))\Gamma\Phi(x(t))\right]^{-1}\left[\Phi^\top(x(t))\Gamma\Phi(x(t))-\gamma\right]dt = \\ +2e^\top(\bar{t})PB\tilde{\Theta}^\top(\bar{t})\Phi(x(\bar{t}))\left[\Phi^\top(x(\bar{t}))\Gamma\Phi(x(\bar{t}))\right]^{-1}\left[\int_{t_i-T_0}^{t_i}\Phi^\top(x(t))\Gamma\Phi(x(t))dt-\gamma T_0\right]=0 \tag{22}$$

Hence, (13) is thus obtained. Then it follows that

$$V_l(t_i)-V_l(t_i-T_0)=V_g(t_i)-V_g(t_i-T_0)\leq 0 \tag{23}$$

Thus, the local Lyapunov candidate function $V_l(t)$ is a piecewise approximation of the global Lyapunov candidate function $V_g(t)$ where their values are equal at the beginning and end points of a time window.

Using the bounded linear approximation of the adaptive law (9), one gets a piecewise locally bounded linear approximation of the standard MRAC adaptive law (9)

$$\dot{u}_{ad}(t)=\dot{\Theta}^\top(t)\Phi(x(t))+\Theta^\top(t)\dot{\Phi}(x(t))\approx-\gamma B^\top Pe(t)+\Theta^\top(t)\dot{\Phi}(x(t)) \tag{24}$$

for $t\in[t_i-T_0,t_i)$, where $t_0=0$, $t_i=t_{i-1}+T_0$, and $i=1,2,\ldots,n\to\infty$.

The second term in the right hand side can be locally approximated by a first-order Taylor series as

$$\Theta^\top(t)\dot{\Phi}(x(t))=\Theta_i^\top\sum_{j=1}^n\frac{\partial\Phi(x(t_i))}{\partial x_j}\dot{x}_j(t_i)+\Theta_i^\top\sum_{j=1}^n\sum_{k=1}^n\frac{\partial^2\Phi(x(t_i))}{\partial x_j\partial x_k}\dot{x}_j(t_i)\left[x_k(t)-x_k(t_i)\right] \\ +\Theta_i^\top\sum_{j=1}^n\frac{\partial\Phi(x(t_i))}{\partial x_j}\left[\dot{x}_j(t)-\dot{x}_j(t_i)\right]+\ldots \\ =\Theta_i^\top\Phi_i'\left[\dot{x}_m(t)-\dot{e}(t)\right]+\Theta_i^\top\Phi_i''\dot{x}_i\left[x_m(t)-e(t)-x_i\right]+\ldots \tag{25}$$

where $\Theta_i=\Theta(t_i)$, $\left(\Phi_i'\right)_j=\partial\Phi(x(t_i))/\partial x_j$ and $\left(\Phi_i''\dot{x}_i\right)_j=\sum_{k=1}^n\dot{x}_k(t_i)\partial^2\Phi(x(t_i))/\partial x_j\partial x_k$, $j=1,\ldots,n$.

Ignoring higher-order terms, the locally bounded linear approximation of the error equation and the standard MRAC adaptive law are expressed as

$$\dot{e}(t)=Ae(t)-BK_xe(t-t_d)+Bu_{ad}(t-t_d)-B\Theta^{*\top}\Phi_i-B\Theta^{*\top}\Phi_i'\left[x_m(t)-e(t)-x_i\right]-BK_x\left[x_m(t)-x_m(t-t_d)\right] \\ +BK_r\left[r(t)-r(t-t_d)\right] \tag{26}$$

$$\dot{u}_{ad}(t)=-\gamma B^\top Pe(t)+\Theta_i^\top\Phi_i'\left[\dot{x}_m(t)-\dot{e}(t)\right] \tag{27}$$

for $t\in[t_i-T_0,t_i)$, where $t_0=0$, $t_i=t_{i-1}+T_0$ and $i=1,2,\ldots,n\to\infty$.

Equations (26) and (27) show that the stability of the locally bounded linear approximation depends on several factors:

- The initial condition $x(0)$
- The structure of the matched uncertainty $\Phi(x(t))$
- The parametric uncertainty Θ^*
- The plant model matrices A, B, A_m, and B_m
- The reference model $x_m(t)$
- The input function $r(t)$
- The adaptive gain parameter γ which includes the adaptive gain Γ as well as the square of the amplitude of $\Phi(x(t))$

Thus, it can be seen that this bounded linear approximation appears to capture the complex nature of stability of a nonlinear adaptive control system, at least in a local sense.

In a special case when $\Phi(x(t)) = x(t)$, then the bounded linear approximation of the error equation and the standard MRAC adaptive law become

$$\dot{e}(t) = Ae(t) - BK_x e(t-t_d) + Bu_{ad}(t-t_d) - B\Theta^{*\top}[x_m(t) - e(t)] - BK_x[x_m(t) - x_m(t-t_d)] + BK_r[r(t) - r(t-t_d)] \tag{28}$$

$$\dot{u}_{ad}(t) = -\gamma B^\top Pe(t) + \Theta_i^\top [\dot{x}_m(t) - \dot{e}(t)] \tag{29}$$

3 Time Delay Margin Estimation of LTI Systems

Consider an input delay closed-loop LTI system

$$\dot{x}(t) = Ax(t) - BKx(t-t_d) \tag{30}$$

where $x(t) : [0,\infty) \to \mathbb{R}^n$ and $\lambda(A - BK) \in \mathbb{C}^-$, i.e., $A - BK$ is Hurwitz.

The time delay margin is defined by the following characteristic equation

$$\det\left(j\omega I - A + BKe^{-j\omega t_d}\right) = 0 \tag{31}$$

For simple systems, analytical solutions of t_d can be computed, but in general such solutions are not easily obtained. We present three methods for estimating the time delay margin.

3.1 Padé Approximation

The Laplace transform of the input delay LTI system is

$$sX(s) - x(0) = AX(s) - BKX(s)e^{-t_d s} \tag{32}$$

Consider the following first-order Padé approximation

$$e^{-t_d s} \approx \frac{2 - t_d s}{2 + t_d s} \tag{33}$$

Then the approximate input delay system becomes

$$(2 + t_d s)s[X(s) - x(0)] = (2 + t_d s)AX(s) - (2 - t_d s)BKX(s) \tag{34}$$

In the time domain, this is expressed as

$$t_d\ddot{x}(t) = (-2I + t_d A + t_d BK)\dot{x}(t) + 2(A - BK)x(t) \tag{35}$$

The time delay margin is then found by

$$\det\left[\omega^2 I + j\omega\left(-\frac{2}{t_d}I + A + BK\right) + \frac{2}{t_d}(A - BK)\right] = 0 \tag{36}$$

Alternatively, the time delay margin can also be obtained as

$$\det\left[\begin{array}{cc} j\omega I & -I \\ -\frac{2}{t_d}(A-BK) & j\omega I+\frac{2}{t_d}I-A-BK \end{array}\right]=0 \tag{37}$$

<u>Example:</u> Given

$$A=\left[\begin{array}{cc} 0 & 1 \\ -1 & 1 \end{array}\right],\ BK=\left[\begin{array}{cc} 0 & 0 \\ 0 & 2 \end{array}\right]$$

The time delay margin using the first-order Padé approximation is estimated to be $t_d = 0.528$ sec.
The exact results can be determined from

$$\det\left[\begin{array}{cc} j\omega & -1 \\ 1 & j\omega-1+2\left(\cos\omega t_d^*-j\sin\omega t_d^*\right) \end{array}\right]=-\omega^2+2\omega\sin\omega t_d^*+1-j\omega\left(1-2\cos\omega t_d^*\right)=0$$

$$\omega=\frac{\sqrt{3}+\sqrt{7}}{2}=2.189\text{ rad/sec}$$

$$t_d=\frac{2\pi}{3\left(\sqrt{3}+\sqrt{7}\right)}=0.478\text{ sec}$$

The effective phase margin is obtained as

$$\phi=\omega t_d=\frac{\pi}{3}$$

So the time delay margin by Padé approximation is non-conservative.

3.2 Lyapunov-Krasovskii Method with Sum-of-Squares Optimization

Stability of time-delay differential equations based on the Lyapunov-Krasovskii method using Lyapunov-Krasovskii functionals have been studied exhaustively [11, 12]. The negative-definiteness of the time derivative of a Lyapunov-Krasovskii functional results in a linear matrix inequality that can be solved for a time delay margin. The solution is generally non-unique since it depends on the choice of a Lyapunov-Krasovskii functional. As a result, the time delay margin obtained by the Lyapunov-Krasovskii method can be conservative.

Theorem 2: For the closed-loop input delay system (30), the system is asymptotically stable if the following linear matrix inequality is satisfied:

$$(A-BK)^\top P+P(A-BK)+(\alpha+\beta)t_dPBKK^\top B^\top P+\frac{t_d}{\alpha}A^\top A+\frac{t_d}{\beta}K^\top B^\top BK<0 \tag{38}$$

The time delay margin is the largest value that renders the LMI feasible.

Proof: For the input delay LTI system (30), we write

$$\int_{t-t_d}^{t}\dot{x}(\tau)\,d\tau=x(t)-x(t-t_d) \tag{39}$$

Then

$$x(t-t_d)=x(t)-\int_{t-t_d}^{t}Ax(\tau)\,d\tau+\int_{t-t_d}^{t}BKx(\tau-t_d)\,d\tau \tag{40}$$

The input delay system now becomes

$$\dot{x}(t)=(A-BK)x(t)+BK\int_{t-t_d}^{t}Ax(\tau)\,d\tau-BK\int_{t-t_d}^{t}BKx(\tau-t_d)\,d\tau \tag{41}$$

Consider the following Lyapunov-Krasovskii functional [11]

$$V(t)=x^\top(t)Px(t)+\int_{t-t_d}^{t}\int_{\tau}^{t}x^\top(\theta)Qx(\theta)\,d\theta d\tau+\int_{t-t_d}^{t}\int_{\tau-t_d}^{t}x^\top(\theta)Sx(\theta)\,d\theta d\tau>0 \tag{42}$$

where $P = P^\top > 0$.

Evaluating $\dot{V}(t)$ yields

$$\dot{V}(t) = \dot{x}^\top(t) P x(t) + x^\top(t) P \dot{x}(t) + \int_{t-t_d}^{t} x^\top(t) Q x(t) d\tau - \int_{t-t_d}^{t} x^\top(\tau) Q x(\tau) d\tau + \int_{t-t_d}^{t} x^\top(t-t_d) S x(t-t_d) d\tau - \int_{t-t_d}^{t} x^\top(\tau - t_d) S x(\tau - t_d) d\tau \quad (43)$$

This becomes

$$\dot{V}(t) = x^\top(t) \left[(A - BK)^\top P + P(A - BK) \right] x(t) + 2x^\top(t) PBK \int_{t-t_d}^{t} Ax(\tau) d\tau - 2x^\top(t) PBK \int_{t-t_d}^{t} BKx(\tau - t_d) d\tau + t_d x^\top(t) Q x(t) - \int_{t-t_d}^{t} x^\top(\tau) Q x(\tau) d\tau + t_d x^\top(t) S x(t) - \int_{t-t_d}^{t} x^\top(\tau - t_d) S x(\tau - t_d) d\tau \quad (44)$$

From the following quadratic expressions

$$\alpha t_d x^\top(t) PBKK^\top B^\top P x(t) - 2x^\top(t) PBK \int_{t-t_d}^{t} Ax(\tau) d\tau + \frac{1}{\alpha t_d} \left(\int_{t-t_d}^{t} x^\top(\tau) A^\top d\tau \right) \left(\int_{t-t_d}^{t} Ax(\tau) d\tau \right) \geq 0 \quad (45)$$

$$\beta t_d x^\top(t) PBKK^\top B^\top P x(t) + 2x^\top(t) PBK \int_{t-t_d}^{t} BKx(\tau - t_d) d\tau + \frac{1}{\beta t_d} \left(\int_{t-t_d}^{t} x^\top(\tau - t_d) K^\top B^\top d\tau \right) \left(\int_{t-t_d}^{t} BKx(\tau - t_d) d\tau \right) \geq 0 \quad (46)$$

we obtain

$$2x^\top(t) PBK \int_{t-t_d}^{t} Ax(\tau) d\tau \leq \alpha t_d x^\top(t) PBKK^\top B^\top P x(t) + \frac{1}{\alpha t_d} \left(\int_{t-t_d}^{t} x^\top(\tau) A^\top d\tau \right) \left(\int_{t-t_d}^{t} Ax(\tau) d\tau \right) \leq \alpha t_d x^\top(t) PBKK^\top B^\top P x(t) + \frac{1}{\alpha} \int_{t-t_d}^{t} x^\top(\tau) A^\top A x(\tau) d\tau \quad (47)$$

$$-2x^\top(t) PBK \int_{t-t_d}^{t} BKx(\tau - t_d) d\tau \leq \beta t_d x^\top(t) PBKK^\top B^\top P x(t) + \frac{1}{\beta t_d} \left(\int_{t-t_d}^{t} x^\top(\tau - t_d) K^\top B^\top d\tau \right) \left(\int_{t-t_d}^{t} BKx(\tau - t_d) d\tau \right) \leq \beta t_d x^\top(t) PBKK^\top B^\top P x(t) + \frac{1}{\beta} \int_{t-t_d}^{t} x^\top(\tau - t_d) K^\top B^\top BKx(\tau - t_d) d\tau \quad (48)$$

for some $\alpha > 0$ and $\beta > 0$.

Choose $Q = A^\top A / \alpha$ and $S = K^\top B^\top BK / \beta$. Then

$$\dot{V}(t) \leq x^\top(t) \left[(A - BK)^\top P + P(A - BK) \right] x(t) + \alpha t_d x^\top(t) PBKK^\top B^\top P x(t) + \beta t_d x^\top(t) PBKK^\top B^\top P x(t) + \frac{t_d}{\alpha} x^\top(t) A^\top A x(t) + \frac{t_d}{\beta} x^\top(t) K^\top B^\top BKx(t) \quad (49)$$

For stability, $\dot{V}(t) < 0$. Thus we obtain the LMI (38).

Example: For the previous example in Section 3.1

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \; BK = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

Choose $\alpha = 1$ and $\beta = 1$. Let

$$(A - BK)^\top P + P(A - BK) = -Q$$

and choose $Q = I$ where I is the identity matrix. Then the time delay margin is estimated to be $t_d = 0.068$ sec.

t_d can be maximized by a suitable selection of α and β. Figure 1 is a plot of t_d as a function of α and β. The maximum value of t_d is 0.072 sec corresponding to $\alpha = 0.51$ and $\beta = 0.89$. As expected, the Lyapunov-Krasovskii method produces a very conservative estimation of the time delay margin.

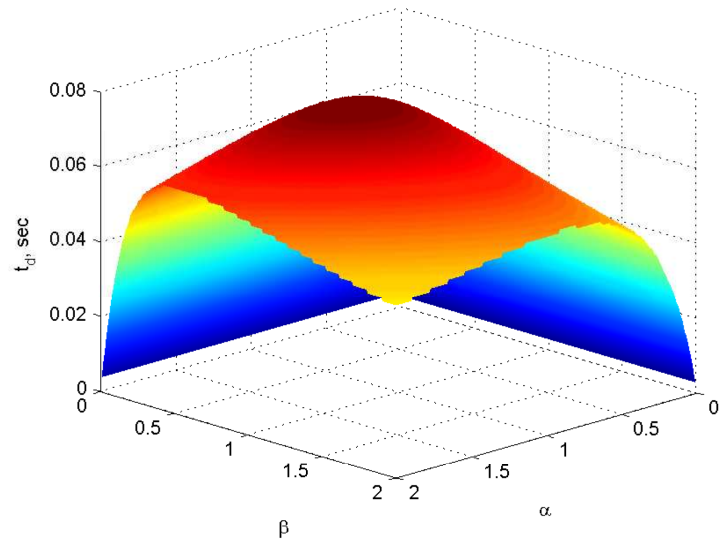

Fig. 1 - Time Delay Margin by Lyapunov-Krasovskii Method

The time delay margin achieved by the Lyapunov-Krasovskii method is directly dependent on the choice of a Lyapunov-Krasovskii functional, which is non-unique. The chosen Lyapunov-Krasovskii functional in Eq. (42) results in a highly conservative estimate of the time delay margin in the example. However, selecting an optimal choice of the Lyapunov-Krasovskii functional by inspection is generally difficult. Thus, sum-of-squares (SOS) optimization is a method that can be employed to search over a specific class of Lyapunov-Krasovskii functionals [13] to improve the estimation of the time delay margin.

A polynomial p is a sum-of-squares (SOS) if there exist polynomials $\{f_i\}_{i=1}^m$ such that

$$p = \sum_{i=1}^m f_i^2$$

For example the polynomial $p = x^2 - 3xy + 14y^2$ is a SOS since $p = (x - 3y)^2 + 5y^2$. All SOS polynomials are positive semi-definite. However, the converse is not true as seen by the Motzkin polynomial $p = x^2y^4 + x^4y^2 + 1 - 3x^2y^2$, which is positive semi-definite, but is not a SOS. Hence, it is apparent that this technique is useful for constructing polynomial Lyapunov functions.

If p is a polynomial of degree less than $2d$ in n variables, then there exists a $Q = Q^\top \geq 0 \in \mathbb{R}^{l_z \times l_z}$ such that $p = z^\top(x) Qz$ where

$$z(x) = \begin{bmatrix} 1 & x_1 & x_2 & \dots & x_n & x_1^2 & x_1x_2 & \dots & x_n^2 & \dots & x_n^d \end{bmatrix}^\top \tag{50}$$

with $z \in \mathbb{R}^{l_z}$ and $l_z = \binom{n+d}{d}$.

This is known as the Gram matrix representation of a SOS. By representing a SOS in the Gram matrix form, the challenge of finding a SOS representation reduces to an LMI. The advantage of the Gram matrix representation of the polynomial is that p is a SOS if and only if there exists $Q \geq 0$ such that $p = z^\top(x) Qz(x)$. However, the Gram matrix Q is generally non-unique.

Naturally, SOS is a useful tool to search over polynomial Lyapunov functions for systems with polynomial dynamics [14]. In particular, we can recast the Lyapunov-Krasovskii stability condition as a SOS program and search over all possible polynomials to find an optimal Lyapunov-Krasovskii functional [13] that can give the best estimate of the time delay margin. Consider the alternate Lyapunov-Krasovskii functional

$$V(x(t_m)) = V_0(x(t)) + \int_{-t_d}^{0} V_1(\tau, x(t), x(t+\tau))\, d\tau + \int_{-t_d}^{0}\int_{t+\tau}^{t} V_2(x(\theta))\, d\theta d\tau \tag{51}$$

where $t_m \in [t-t_d, t]$, and V_0 , V_1, and V_2 are polynomials to be optimized.

Thus, we can search over all possible polynomials V_0, V_1, and V_2 to maximize the time delay margin. The search method involves invoking the following lemma [13]:

Lemma 1: Let $\dot{x}(t) = f(x(t), x(t-t_d))$ and assume that the origin is an equilibrium point. Assume that there exist V_0, V_1, and V_2 and polynomial $\psi(x(t)) > 0$ such that the following conditions are satisfied:

1. $V_0(x(t)) - \psi(x(t)) \geq 0$
2. $V_1(\tau, x(t), x(t+\tau)) \geq 0 \forall \tau \in [-t_d, 0]$
3. $V_2(x(\theta)) \geq 0$
4. $\dot{V}(x(t_m)) = \frac{dV_0}{dx(t)} f(x(t), x(t-t_d)) + t_d \frac{\partial V_1}{\partial x(t)} f(x(t), x(t-t_d)) - r\frac{\partial V_1}{\partial \tau} + V_1(0, x(t), x(t)) - V_1(-t_d, x(t), x(t-t_d)) + t_d V_2(x(t)) - t_d V_2(x(t+\tau)) \leq 0 \forall \tau \in [-t_d, 0]$

Then the origin is a stable equilibrium for all time delays in $[0, t_d]$.

Condition 1 simply ensures that V_0 is positive definite. Conditions 2 and 3 requires V_1 and V_2 to be positive semi-definite on appropriate intervals. Thus, V will be positive definite. Lastly, condition 4 guarantees that $\dot{V} \leq 0$. Hence, if Lemma 1 holds, then the Lyapunov-Krasovskii functional in Eq. (51) certifies the stability of the system $\dot{x}(t) = f(x(t), x(t-t_d))$ with $f(0,0) = 0$ for time delays up to t_d.

Example: For the previous example, the SOS optimization of V_0, V_1, and V_2 is performed using the freely available software SOSOPT [15]. The resulting polynomials are given by

$$V_0(x(t)) = 500.3x_1^2(t) + 344.3x_1(t)x_2(t) + 61.83x_2^2(t)$$

$$\begin{aligned} V_1(x(t), x(t+\tau)) = {} & 660.9x_1^2(t) - 267.4x_1(t)x_2(t) - 46.71x_1(t)x_1(t+\tau) + 435.9x_1(t)x_2(t+\tau) + 3273x_2^2(t) \\ & - 1.972x_2(t)x_1(t+\tau) - 4037x_2(t)x_2(t+\tau) + 27.82x_1^2(t+\tau) - 4.292x_1(t+\tau)x_2(t+\tau) + 1280x_2^2(t+\tau) \end{aligned}$$

$$V_2(x(\theta)) = 40.76x_1^2(\theta) + 23x_1(\theta)x_2(\theta) + 3307x_2^2(\theta)$$

The time delay margin for the example is $t_d = 0.216$ sec which is three times greater than the previous result using the Lyapunov-Krasovskii functional in Eq. (42).

The challenge using a SOS optimization is that the problem can quickly become intractable as the number of states n or the degree of the polynomial $2d$ increases. However, this method is extremely useful on modest-sized problems.

3.3 Matrix Measure Method

The matrix measure method has been developed recently and affords a simple way to estimate the time delay margin and the effective phase margin for the MIMO LTI system [1]. The matrix measure μ is defined as an eigenvalue of a symmetric part of a complex matrix [11] such that

$$\mu_i(C) = \lambda_i\left(\frac{C+C^*}{2}\right) \tag{52}$$

where $C \in \mathbb{C}$ is a complex matrix and C^* is its complex conjugate transpose, then μ has the following properties

$$\mu_i(C) \in \mathbb{R} \tag{53}$$

$$\overline{\mu}(C) = \max_{1\leq i\leq n} \lambda_i\left(\frac{C+C^*}{2}\right) = \lim_{\varepsilon\to 0} \frac{\|I+\varepsilon C\| - 1}{\varepsilon} \tag{54}$$

$$\underline{\mu}(C) = \min_{1\le i\le n} \lambda_i \left(\frac{C+C^*}{2}\right) = \lim_{\varepsilon\to 0} \frac{1-\|I-\varepsilon C\|}{\varepsilon} \tag{55}$$

$$\underline{\mu}(C) \le \mathrm{Re}\lambda_i(C) \le \overline{\mu}(C) \tag{56}$$

$$\mathrm{Im}\lambda(C) \le \overline{\mu}(-jC) \tag{57}$$

Theorem 3: The input delay LTI system (30) is asymptotically stable if the following inequalities hold

$$t_d \le \frac{1}{\omega}\cos^{-1}\left[\frac{\overline{\mu}(A)+\overline{\mu}(jBK)}{\|BK\|}\right] \tag{58}$$

$$\omega = \overline{\mu}(-jA) + \|BK\| \tag{59}$$

where $\|.\| = \|.\|_2$ is the $\mathscr{L}_2$-norm.

Proof: The real parts of the system poles are bounded from above by

$$\sigma = \mathrm{Re}\lambda\left(A - BKe^{-j\omega t_d}\right) \le \overline{\mu}(A) + \overline{\mu}\left(-BKe^{-j\omega t_d}\right) \le \overline{\mu}(A) + \overline{\mu}(-BK)|\cos\omega t_d| + \overline{\mu}(jBK)|\sin\omega t_d| \tag{60}$$

Let $0 \le \omega t_d \le \frac{\pi}{2}$, then the input delay system is stable if $\sigma \le 0$ which implies

$$\begin{aligned}\overline{\mu}(A) \le -\overline{\mu}(-BK)\cos\omega t_d - \overline{\mu}(jBK)\sin\omega t_d &= \underline{\mu}(BK)\cos\omega t_d - \overline{\mu}(jBK)\sin\omega t_d \\ &\le \overline{\mu}(BK)\cos\omega t_d - \overline{\mu}(jBK)\sin\omega t_d\end{aligned} \tag{61}$$

Upon some algebra, this can be expressed as

$$\left[\overline{\mu}^2(BK) + \overline{\mu}^2(jBK)\right]\cos^2\omega t_d - 2\overline{\mu}(A)\overline{\mu}(BK)\cos\omega t_d + \overline{\mu}^2(A) - \overline{\mu}^2(jBK) \ge 0 \tag{62}$$

The solution yields a bound on time delay margin t_d as

$$t_d \le \frac{1}{\omega}\cos^{-1}\frac{\overline{\mu}(A)\overline{\mu}(BK) + \overline{\mu}(jBK)\sqrt{\overline{\mu}^2(BK)+\overline{\mu}^2(jBK)-\overline{\mu}^2(A)}}{\overline{\mu}^2(BK)+\overline{\mu}^2(jBK)} \tag{63}$$

But

$$\overline{\mu}^2(BK) \le \overline{\mu}^2(BK) + \overline{\mu}^2(jBK) \le \|BK\|^2 \tag{64}$$

So

$$t_d \le \frac{1}{\omega}\cos^{-1}\left[\frac{\overline{\mu}(A)\|BK\| + \overline{\mu}(jBK)\sqrt{\|BK\|^2-\overline{\mu}^2(A)}}{\|BK\|^2}\right] \le \frac{1}{\omega}\cos^{-1}\left[\frac{\overline{\mu}(A)+\overline{\mu}(jBK)}{\|BK\|}\right] \tag{65}$$

The imaginary parts of the system poles are bounded from above by

$$\omega = \mathrm{Im}\lambda\left(-jA + jBKe^{-j\omega t_d}\right) \le \overline{\mu}(-jA) + \overline{\mu}\left(jBKe^{-j\omega t_d}\right) \le \overline{\mu}(-jA) + \overline{\mu}(jBK)|\cos\omega t_d| + \overline{\mu}(BK)|\sin\omega t_d| \tag{66}$$

which can be expressed as

$$\omega \le \overline{\mu}(-jA) + \sqrt{\overline{\mu}^2(BK)+\overline{\mu}^2(jBK)} \le \overline{\mu}(-jA) + \|BK\| \tag{67}$$

Since t_d must be the smallest value for all permissible values of ω, therefore the equality sign applies. Thus

$$\omega = \overline{\mu}(-jA) + \sqrt{\overline{\mu}^2(BK)+\overline{\mu}^2(jBK)} \le \overline{\mu}(-jA) + \|BK\| \tag{68}$$

Example: For the previous example

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \; BK = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

ω and t_d are computed as follows:

$$\overline{\mu}(A) = 1$$

$$\overline{\mu}\left(jBK\right)=0$$

$$\|BK\|=2$$

$$\omega=\overline{\mu}\left(-jA\right)+\|BK\|=3\text{ rad/sec}$$

$$t_d=\frac{1}{\omega}\cos^{-1}\left[\frac{\overline{\mu}\left(A\right)}{\|BK\|}\right]=\frac{\pi}{9}=0.349\text{ sec}$$

Comparing the result with the exact time delay margin, the matrix measure method produces the least conservative estimation of the time delay margin. Moreover, the solution is much simpler to compute than both the Padé approximation and Lyapunov-Krasovskii method.

The matrix measure method can also estimate the effective phase margin of a MIMO system. For this example, the phase margin estimate is the same as the exact value.

In summary, the comparison among the methods presented is shown in Table 1.

Method	t_d, sec
Padé Approximation	0.528
Lyapunov-Krasovskii $\alpha=\beta=1$	0.072
Lyapunov-Krasovskii with SOS Optimization	0.216
Matrix Measure	0.349
Exact Value	0.478

Table 1 - Comparison of Time Delay Margin Estimates of LTI Example

4 Time Delay Margin Estimation of Adaptive Control

4.1 Standard MRAC

Using the bounded linear stability analysis method to approximate the standard MRAC as a bounded linear approximation within a given time window, the time delay margin of adaptive control can be estimated by any of the methods previously presented. Differentiating the error equation and substituting in the adaptive law which yield

$$\ddot{e}\left(t\right)=\left(A+B\Theta^{*\top}\Phi_i'\right)\dot{e}\left(t\right)-\left(BK_x+B\Theta_i^{\top}\Phi_i'\right)\dot{e}\left(t-t_d\right)-\gamma BB^{\top}Pe\left(t-t_d\right)+B\Theta_i^{\top}\Phi_i'\dot{x}_m\left(t-t_d\right)$$
$$-B\Theta^{*\top}\Phi_i'\dot{x}_m\left(t\right)-BK_x\left[\dot{x}_m\left(t\right)-\dot{x}_m\left(t-t_d\right)\right]+BK_r\left[\dot{r}\left(t\right)-\dot{r}\left(t-t_d\right)\right] \quad (69)$$

for $t\in[t_i-T_0,t_i)$, where $t_0=0$, $t_i=t_{i-1}+T_0$ and $i=1,2,\dots,n\to\infty$.

To use the previous results, the problem is restricted to the case when the reference model is zero or with a step input reference command signal. Let $r\left(t\right)$ be a constant signal, then $\dot{r}\left(t\right)=\dot{r}\left(t-t_d\right)=0$. Then $\dot{x}_m\left(t\right)=\dot{x}_m\left(t-t_d\right)=0$ after some time $t=t_0>0$. Then the error equation can be recast as

$$\dot{z}\left(t\right)=C_iz\left(t\right)-D_iz\left(t-t_d\right) \quad (70)$$

for $t\in[t_i-T_0,t_i)$, where $t_i=t_{i-1}+T_0$ and $i=1,2,\dots,n\to\infty$, and $z\left(t\right)=\left[\begin{array}{cc} e\left(t\right) & \dot{e}\left(t\right)\end{array}\right]$ and

$$C_i=\left[\begin{array}{cc} 0 & I \\ 0 & A+B\Theta^{*\top}\Phi_i' \end{array}\right] \quad (71)$$

$$D_i=\left[\begin{array}{cc} 0 & 0 \\ \gamma BB^{\top}P & BK_x+B\Theta_i^{\top}\Phi_i' \end{array}\right] \quad (72)$$

1. Padé approximation:

   Using the result in the previous section, the time delay margin can be found by solving the following characteristic equation

$$\det\left[\begin{array}{cc} j\omega_iI & -I \\ -\frac{2}{t_{d_i}}\left(C_i-D_i\right) & j\omega_i+\frac{2}{t_d}I-C_i-D_i \end{array}\right]=0 \quad (73)$$

Alternatively, the first-order Padé approximation of the bounded linear approximation of the error equations and adaptive law can be expressed as

$$sE(s) - e(0) = \left(A + B\Theta^{*\top}\Phi_i'\right) E(s) - BK_x E(s) \frac{2 - t_d s}{2 + t_d s} + BU_{ad}(s) \frac{2 - t_d s}{2 + t_d s} - B\Theta^{*\top}\Phi_i - B\Theta^{*\top}\Phi_i' \left[X_m(s) - x_i\right]$$
$$- BK_x \left[X_m(s) - X_m(s) \frac{2 - t_d s}{2 + t_d s}\right] + BK_r \left[R(s) - R(s) \frac{2 - t_d s}{2 + t_d s}\right] \quad (74)$$

$$U_{ad}(s) = -\gamma B^\top P \frac{E(s)}{s} + \Theta_i^\top \Phi_i' \left[X_m(s) - E(s)\right] \quad (75)$$

Then the time delay margin can be computed from the following characteristic equation:

$$\det\left[j\omega_i I - \left(A + B\Theta^{*\top}\Phi_i'\right) + \left(BK_x + B\Theta_i^\top \Phi_i' + \frac{\gamma BB^\top P}{j\omega_i}\right) \frac{2 - j\omega_i t_{d_i}}{2 + j\omega_i t_{d_i}}\right] = 0 \quad (76)$$

or equivalently

$$\det \begin{bmatrix} j\omega_i I & -I & 0 \\ -\frac{2}{t_{d_i}}\left(A + B\Theta^{*\top}\Phi_i' - BK_x\right) + \gamma BB^\top P & j\omega_i I + \frac{2}{t_{d_i}} I - A - B\Theta^{*\top}\Phi_i' - BK_x - B\Theta_i^\top \Phi_i' & -\frac{2}{t_{d_i}} B \\ -\gamma B^\top P & \Theta_i^\top \Phi_i' & j\omega_i I \end{bmatrix} = 0 \quad (77)$$

2. Lyapunov-Krasovskii method:

   Applying the result of the Lyapunov-Krasovskii method from the previous section, the time delay margin can be computed from the following LMI:

$$(C_i - D_i)^\top P + P(C_i - D_i) + (\alpha + \beta) t_{d_i} P D_i D_i^\top P + \frac{t_{d_i}}{\alpha} C_i^\top C_i + \frac{t_{d_i}}{\beta} D_i^\top D_i < 0 \quad (78)$$

   The SOS optimization can also be used to obtain a better time delay margin estimate.

3. Matrix measure method:

   Using the matrix measure method, the time delay margin is estimated as

$$t_{d_i} \leq \frac{1}{\omega_i} \cos^{-1}\left[\frac{\overline{\mu}(C_i) + \overline{\mu}(jD_i)}{\|D_i\|}\right] \quad (79)$$

$$\omega_i = \overline{\mu}(-jC_i) + \|D_i\| \quad (80)$$

   Another approach is to derive the time delay margin from the characteristic equation of the bounded linear approximation of the error equation (69)

$$\det\left[j\omega_i I - \left(A + B\Theta^{*\top}\Phi_i'\right) + \left(BK_x + B\Theta_i^\top \Phi_i' + \frac{\gamma BB^\top P}{j\omega_i}\right) e^{-j\omega_i t_{d_i}}\right] = 0 \quad (81)$$

   Then, applying the matrix measure method, the time delay margin is estimated from

$$\sigma \leq \overline{\mu}\left(A + B\Theta^{*\top}\Phi_i'\right) + \overline{\mu}\left[-\left(BK_x + B\Theta_i^\top \Phi_i'\right)\right] \cos\omega_i t_{d_i} + \overline{\mu}\left[j\left(BK_x + B\Theta_i^\top \Phi_i'\right)\right] \sin\omega_i t_{d_i}$$
$$+ \frac{1}{\omega_i} \overline{\mu}\left(j\gamma BB^\top P\right) \cos\omega_i t_{d_i} + \frac{1}{\omega_i} \overline{\mu}\left(\gamma BB^\top P\right) \sin\omega_i t_{d_i} \leq 0 \quad (82)$$

   which yields

$$t_{d_i} \leq \frac{1}{\omega_i} \cos^{-1}\left[\frac{\omega_i \overline{\mu}\left(A + B\Theta^{*\top}\Phi_i'\right)\left[\omega_i \left\|BK_x + B\Theta_i^\top \Phi_i'\right\| - \overline{\mu}\left(j\gamma BB^\top P\right)\right]}{\omega_i^2 \left\|BK_x + B\Theta_i^\top \Phi_i'\right\|^2 + \left\|\gamma BB^\top P\right\|^2}\right.$$
$$\left. + \frac{\left[\omega_i \overline{\mu}\left[j\left(BK_x + B\Theta_i^\top \Phi_i'\right)\right] + \left\|\gamma BB^\top P\right\|\right] \sqrt{\omega_i^2 \left\|BK_x + B\Theta_i^\top \Phi_i'\right\|^2 + \left\|\gamma BB^\top P\right\|^2}}{\omega_i^2 \left\|BK_x + B\Theta_i^\top \Phi_i'\right\|^2 + \left\|\gamma BB^\top P\right\|^2}\right] \quad (83)$$

The frequency is computed from

$$\omega_i \le \overline{\mu}\left[-j\left(A+B\Theta^{*\top}\Phi_i'\right)\right]+\overline{\mu}\left[j\left(BK_x+B\Theta_i^\top\right)e^{-j\omega_i t_{d_i}}\right]+\frac{1}{\omega_i}\overline{\mu}\left(\gamma BB^\top Pe^{-j\omega_i t_{d_i}}\right)$$
$$\le \overline{\mu}\left[-j\left(A+B\Theta^{*\top}\Phi_i'\right)\right]+\left\|BK_x+B\Theta_i^\top\right\|+\frac{1}{\omega_i}\left\|\gamma BB^\top P\right\| \quad (84)$$

which yields the value of ω_i that renders t_d a minimum

$$\omega_i = \frac{\overline{\mu}\left[-j\left(A+B\Theta^{*\top}\Phi_i'\right)\right]+\left\|BK_x+B\Theta_i^\top\right\|}{2}\left(1+\sqrt{1+\frac{4\left\|\gamma BB^\top P\right\|}{\left\{\overline{\mu}\left[-j\left(A+B\Theta^{*\top}\Phi_i'\right)\right]+\left\|BK_x+B\Theta_i^\top\right\|\right\}^2}}\right) \quad (85)$$

It is noted that $\omega_i \to \infty$ as $\gamma \to \infty$. Consequently, the time delay margin tends to zero as $\Gamma \to \infty$. This is consistent with the behavior of the standard MRAC.

4.2 Scalar MRAC

Consider an input delay scalar MRAC system with linear structured uncertainty

$$\dot{x}(t) = ax(t) + b\left[u(t-t_d)+\theta^* x(t)\right] \quad (86)$$

The reference model is given by

$$\dot{x}_m(t) = a_m x_m(t) + b_m r(t) \quad (87)$$

The controller is given by

$$u(t) = -k_x x(t) + k_r r(t) - \theta(t)x(t) \quad (88)$$

$$\dot{\theta}(t) = -\Gamma x(t)\, pbe(t) \quad (89)$$

1. The time delay margin is estimated from the matrices C_i and D_i

$$C_i = \begin{bmatrix} 0 & 1 \\ 0 & a+b\theta^* \end{bmatrix} \quad (90)$$

$$D_i = \begin{bmatrix} 0 & 0 \\ \gamma b^2 p & bk_x + b\theta_i \end{bmatrix} \quad (91)$$

where

$$\gamma = \frac{\Gamma}{T_0}\int_{t_i-T_0}^{t_i} x^2(\tau)\,d\tau \quad (92)$$

Using the matrix measure method, the following parameters are computed analytically as

$$\overline{\mu}(C_i) = \frac{a+b\theta^*+\sqrt{(a+b\theta^*)^2+1}}{2} \quad (93)$$

$$\overline{\mu}(-jC_i) = \frac{1}{2} \quad (94)$$

$$\overline{\mu}(jD_i) = \frac{\gamma b^2 p}{2} \quad (95)$$

$$\|D_i\| = \sqrt{(bk_x+b\theta_i)^2+\gamma^2 b^4 p^2} \quad (96)$$

ω_i and t_{d_i} are then estimated as

$$\omega_i = \frac{1}{2}+\sqrt{(bk_x+b\theta_i)^2+\gamma^2 b^4 p^2} \quad (97)$$

$$t_{d_i} = \frac{1}{\omega_i}\cos^{-1}\left[\frac{a+b\theta^*+\sqrt{(a+b\theta^*)^2+1}+\gamma b^2 p}{2\sqrt{(bk_x+b\theta_i)^2+\gamma^2 b^4 p^2}}\right] \quad (98)$$

2. Using the original system parameters, the time delay margin is estimated as

$$\omega_i = \frac{|bk_x + b\theta_i|}{2} + \frac{1}{2}\sqrt{(bk_x + b\theta_i)^2 + 4\gamma b^2 p} \tag{99}$$

$$t_{d_i} = \frac{1}{\omega_i}\cos^{-1}\left[\frac{\omega_i^2 (a + b\theta^*)|bk_x + b\theta_i| + \gamma b^2 p\sqrt{\omega_i^2 (bk_x + b\theta_i)^2 + \gamma^2 b^4 p^2}}{\omega_i^2 (bk_x + b\theta_i)^2 + \gamma^2 b^4 p^2}\right] \tag{100}$$

Both approaches yield somewhat different results. The "exact" values of ω_i and t_{d_i} for the locally bounded linear approximation of the error equation can be determined as follows:

$$\det\left[j\omega - (a + b\theta^*) + (bk_x + b\theta_i)e^{-j\omega t_d} + \gamma b^2 p\frac{e^{-j\omega t_d}}{j\omega}\right] = 0 \tag{101}$$

This results in two equations

$$-\omega_i^2 + (bk_x + b\theta_i)\omega_i \sin\omega_i t_{d_i} + \gamma b^2 p\cos\omega_i t_{d_i} = 0 \tag{102}$$

$$-(a + b\theta^*)\omega_i + (bk_x + b\theta_i)\omega_i \cos\omega_i t_{d_i} - \gamma b^2 p\sin\omega_i t_{d_i} = 0 \tag{103}$$

The frequency equation is obtained as

$$\omega_i^4 + \left[(a + b\theta^*)^2 - (bk_x + b\theta_i)^2\right]\omega_i^2 - \gamma^2 b^4 p^2 = 0 \tag{104}$$

The "exact" solution gives

$$\omega_i = \sqrt{\frac{(bk_x + b\theta_i)^2 - (a + b\theta^*)^2 + \sqrt{\left[(a + b\theta^*)^2 - (bk_x + b\theta_i)^2\right]^2 + 4\gamma^2 b^4 p^2}}{2}} \tag{105}$$

$$t_{d_i} = \frac{1}{\omega_i}\cos^{-1}\left[\frac{\omega_i^2 (a + b\theta^*)(bk_x + b\theta_i) + \omega_i^2 \gamma b^2 p}{\omega_i^2 (bk_x + b\theta_i)^2 + \gamma^2 b^4 p^2}\right] \tag{106}$$

It is well-known that $t_d \to 0$ as as $\Gamma \to \infty$ for the standard MRAC. This behavior is exhibited in the time delay margin estimation by the matrix measure method and the "exact" solution since

$$\lim_{\gamma\to\infty}\frac{1}{\omega_i} = \lim_{\gamma\to\infty}\frac{1}{\sqrt{\gamma b^2 p}} = 0 \tag{107}$$

For $\gamma = 0$, the system is non-adaptive and the time delay margin estimation by the matrix measure method using the C_i and D_i yields

$$t_d = \frac{2}{1 + 2bk_x}\cos^{-1}\left[\frac{a + b\theta^* + \sqrt{(a + b\theta^*)^2 + 1}}{2bk_x}\right] \tag{108}$$

The exact time delay margin is computed to be

$$t_d = \frac{1}{\sqrt{(bk_x)^2 - (a + b\theta^*)^2}}\cos^{-1}\left(\frac{a + b\theta^*}{bk_x}\right) \tag{109}$$

<u>Example:</u> Given $a = 1$, $b = 1$, $\theta^* = 0.1$, $a_m = -1$, $b_m = 1$, $p = 1$, $\theta(0) = 0$, $r(t) = 1$. The control gains are computed to be $k_x = 2$ and $k_r = 1$. The adaptive gain is selected as $\Gamma = 1$.

For the non-adaptive LTI system for which $\theta(t) = 0$ for all t, the time delay margin estimates and the exact value are shown in Table 2.

Method	t_d, sec
Padé Approximation	0.646
Lyapunov-Krasovskii $\alpha = \beta = 1$	0.137
Lyapunov-Krasovskii with SOS Optimization	0.235
Matrix Measure in Part 1	0.347
Matrix Measure in Part 2	0.494
Exact Value	0.592

Table 2 - Time Delay Margin Estimation of Non-Adaptive System

Thus, it can be seen that the time delay margin computed by the matrix measure method is the least conservative lower bound estimate of the true time delay margin among the present approaches. The difference in both approaches using the matrix measure method is noted. The approach using the matrices C_i and D_i are more conservative.

Figure 2 is a plot of the variation of the local time delay margin estimates computed by the matrix measure method using the matrices C_i and D_i within three different time windows with $T_0 = 1$ sec, $T_0 = 5$ sec, and $T_0 = 10$ sec based on the bounded linear stability analysis method. It is noted that as the window size increases, the initial transients in the time delay margin estimates tend to decrease. However, the estimates do converge to a constant value regardless of the time window sizes. In a previous study, it was found that the mean value of the computed local time delay margins is relatively insensitive to the window size.

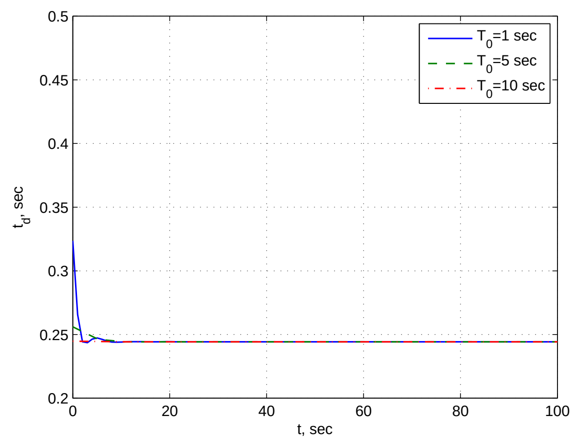

Fig. 2 - Time Delay Margin Estimates

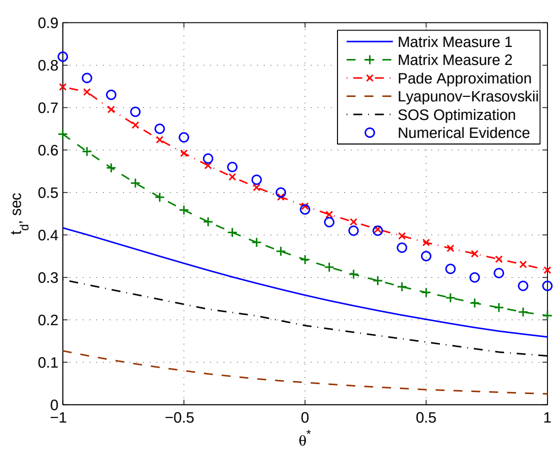

Fig. 3 - Time Delay Margin Variation with Unknown Parameter θ^*

Figure 3 is a plot of the mean value of the time delay margin estimates by all the different methods as a function of the unknown parameter $-1 \leq \theta^* \leq 1$ for $T_0 = 1$ sec. Generally, θ^* is not known, so in a verification setting, the time delay margin should be computed over all possible parameter variations within their physical bounds. Also plotted is the numerical evidence of the time delay margin from simulations. Comparing to the numerical evidence, both the time delay margin estimates by the matrix measure method using the matrices C_i and D_i and the original system parameters are reasonably conservative. The matrix measure method using the original system parameters is able to estimate the time delay margin better than the same method but using the matrices C_i and D_i. The Padé approximation gives the best estimate of the time delay margin, but is non-conservative since it over-estimates the time delay margin for $\theta^* > 0$. The Lyapunov-Krasovskii method, as expected, yields the most conservative estimates of the time delay margin. The difference ranges from about 7 times smaller for $\theta^* = -1$ to 11 times smaller $\theta^* = 1$. However, with the SOS optimization of the Lyapunov-Krasovskii functional, the time delay margin estimates are improved considerably. Thus, in summary, the matrix measure method using the original system parameters appears to produce more reasonably conservative estimation of the time delay margin among all the methods.

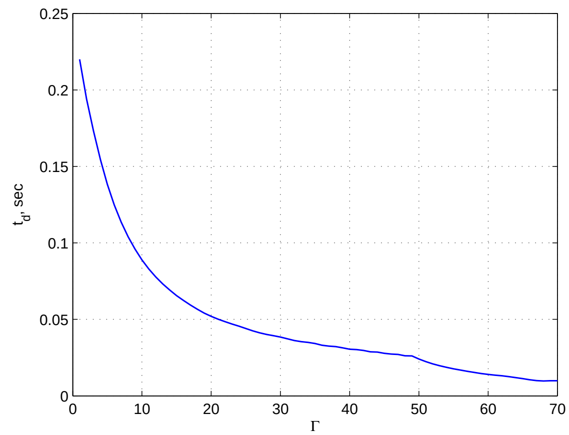

Fig. 4 - Time Delay Margin Variation with Adaptive Gain

Figure 4 is a plot of the mean value of the time delay margin estimates by the matrix measure method using the matrices C_i and D_i as a function of the adaptive gain $1 \leq \Gamma \leq 100$ for $T_0 = 1$ sec. It can be seen that as the adaptive gain Γ increases, the time delay margin of the adaptive system decreases as expected for the standard MRAC.

4.3 σ-Modification

The σ-modification adaptive law [9] is given by

$$\dot{\Theta}(t) = -\Gamma\Phi(x(t))\left[e^\top(t)PB + \sigma\Theta(t)\right] \tag{110}$$

Using the bounded linear stability analysis method, the adaptive law is approximated as

$$\dot{u}_{ad}(t) = -\gamma B^\top Pe(t) - \underline{\Gamma}\sigma u_{ad}(t) + \Theta_i^\top \Phi_i' \left[\dot{x}_m(t) - \dot{e}(t)\right] \tag{111}$$

for $t \in [t_i - T_0, t_i)$, where $t_0 = 0$, $t_i = t_{i-1} + T_0$ and $i = 1, 2, \ldots, n \to \infty$, and $\underline{\Gamma} = \lambda_{min}(\Gamma)$.

Then the error equation is obtained as

$$\begin{aligned}\ddot{e}(t) = {} & \left(A + B\Theta^{*\top}\Phi_i' - \underline{\Gamma}\sigma\right)\dot{e}(t) - \left(BK_x + B\Theta_i^\top\Phi_i'\right)\dot{e}(t - t_d) + \underline{\Gamma}\sigma\left(A + B\Theta^{*\top}\Phi_i'\right)e(t) \\ & - \left(\gamma BB^\top P + \underline{\Gamma}\sigma BK_x\right)e(t - t_d) - \underline{\Gamma}\sigma B\Theta^{*\top}\left[\Phi_i + \Phi_i' x_m(t) - \Phi_i' x_i\right] - \underline{\Gamma}\sigma BK_x\left[x_m(t) - x_m(t - t_d)\right] \\ & + \underline{\Gamma}\sigma BK_r\left[r(t) - r(t - t_d)\right] + B\Theta_i^\top\Phi_i'\dot{x}_m(t - t_d) - B\Theta^{*\top}\Phi_i'\dot{x}_m(t) - BK_x\left[\dot{x}_m(t) - \dot{x}_m(t - t_d)\right] \\ & + BK_r\left[\dot{r}(t) - \dot{r}(t - t_d)\right]\end{aligned} \tag{112}$$

For a step input reference command signal, the error equation becomes

$$\dot{z}(t) = C_i z(t) - D_i z(t - t_d) \tag{113}$$

where $z(t) = \begin{bmatrix} e(t) & \dot{e}(t) \end{bmatrix}$ and

$$C_i = \begin{bmatrix} 0 & I \\ \underline{\Gamma}\sigma\left(A + B\Theta^{*\top}\Phi_i'\right) & A + B\Theta^{*\top}\Phi_i' - \underline{\Gamma}\sigma \end{bmatrix} \tag{114}$$

$$D_i = \begin{bmatrix} 0 & 0 \\ \gamma BB^\top P + \underline{\Gamma}\sigma BK_x & BK_x + B\Theta_i^\top \Phi_i' \end{bmatrix} \tag{115}$$

The time margin for the σ-modification adaptive law can then be estimated by the following methods:

1. Padé Approximation:

   The time delay margin can be found from the following characteristic equation:

$$\det\left[j\omega_i I - \left(A + B\Theta^{*\top}\Phi_i'\right) + \left(BK_x + B\Theta_i^\top \Phi_i' \frac{j\omega_i}{j\omega_i + \underline{\Gamma}\sigma} + \frac{\gamma BB^\top P}{j\omega_i + \underline{\Gamma}\sigma}\right)\frac{2 - j\omega_i t_{d_i}}{2 + j\omega_i t_{d_i}}\right] = 0 \tag{116}$$

   or equivalently

$$\det\begin{bmatrix} j\omega_i I & -I & 0 \\ -\frac{2}{t_{d_i}}\left(A + B\Theta^{*\top}\Phi_i' - BK_x\right) + \gamma BB^\top P & j\omega_i I + \frac{2}{t_{d_i}} I - A - B\Theta^{*\top}\Phi_i' - BK_x - B\Theta_i^\top\Phi_i' & -\frac{2}{t_{d_i}}B - \frac{1}{t_{d_i}}B\underline{\Gamma}\sigma \\ -\gamma B^\top P & \Theta_i^\top \Phi_i' & j\omega_i I + \underline{\Gamma}\sigma \end{bmatrix} = 0 \tag{117}$$

2. Lyapunov-Krasovskii method:

   In theory, the time delay margin could be estimated using Eq. (78) and the matrices C_i and D_i, but the result is expected to be extremely conservative so the solution of the LMI may be infeasible. A suitable Lyapunov-Krasovskii method should be considered to account for the σ-modification term in the adaptive law.

3. Matrix measure method:

   The time delay margin may be estimated from Eqs. (79) and (80) using the matrices C_i and D_i. However, it is observed that as $\Gamma \to \infty$, $\|D_i\| \to \infty$ so that $\omega_i \to \infty$ which yields $t_d \to 0$. This is not consistent which the fact that robust modification will result in a finite time delay margin as $\Gamma \to \infty$. Thus, the characteristic equation of the input delay adaptive system

$$\det\left[j\omega_i I - \left(A + B\Theta^{*\top}\Phi_i'\right) + \left(BK_x + B\Theta_i^\top \Phi_i' \frac{j\omega_i}{j\omega_i + \underline{\Gamma}\sigma} + \frac{\gamma BB^\top P}{j\omega_i + \underline{\Gamma}\sigma}\right) e^{-j\omega_i t_{d_i}}\right] = 0 \tag{118}$$

   should be used to compute the time delay margin for the σ-modification adaptive law instead. The characteristic equation can be recast as

$$\det\left\{-\omega_i I + j\underline{\Gamma}\sigma - j\left(A + B\Theta^{*\top}\Phi_i'\right) - \left(A + B\Theta^{*\top}\Phi_i'\right)\frac{\underline{\Gamma}\sigma}{\omega_i} + \left[j\left(BK_x + B\Theta_i^\top\Phi_i'\right) + \frac{1}{\omega_i}\left(\gamma BB^\top P + BK_x\underline{\Gamma}\sigma\right)\right] e^{-j\omega_i t_{d_i}}\right\} = 0 \tag{119}$$

   The time delay margin can be obtained from

$$\begin{aligned}\sigma \le \omega_i + \overline{\mu}\left[j\left(A + B\Theta^{*\top}\Phi_i'\right)\right] + \frac{1}{\omega_i}\overline{\mu}\left[\left(A + B\Theta^{*\top}\Phi_i'\right)\underline{\Gamma}\sigma\right] + \overline{\mu}\left[-j\left(BK_x + B\Theta_i^\top\Phi_i'\right)\right]\cos\omega_i t_{d_i} \\ + \overline{\mu}\left[-\left(BK_x + B\Theta_i^\top\Phi_i'\right)\right]\sin\omega_i t_{d_i} + \frac{1}{\omega_i}\overline{\mu}\left[-\left(\gamma BB^\top P + BK_x\underline{\Gamma}\sigma\right)\right]\cos\omega_i t_{d_i} \\ + \frac{1}{\omega_i}\overline{\mu}\left[j\left(\gamma BB^\top P + BK_x\underline{\Gamma}\sigma\right)\right]\sin\omega_i t_{d_i} \le 0\end{aligned} \tag{120}$$

which yields

$$t_{d_i} \leq \frac{1}{\omega_i} \cos^{-1} \left[\left\{ \omega_i^2 + \omega_i \overline{\mu} \left[j \left(A + B\Theta^{*\top} \Phi_i' \right) \right] + \overline{\mu} \left[\left(A + B\Theta^{*\top} \Phi_i' \right) \underline{\Gamma} \sigma \right] \right\} \times \right.$$
$$\times \frac{\left\{ \left\| \gamma BB^\top P + BK_x \underline{\Gamma} \sigma \right\| - \omega_i \overline{\mu} \left[j \left(BK_x + B\Theta_i^\top \Phi_i' \right) \right] \right\}}{\omega_i^2 \left\| BK_x + B\Theta_i^\top \Phi_i' \right\|^2 + \left\| \gamma BB^\top P + BK_x \underline{\Gamma} \sigma \right\|^2}$$
$$\left. + \frac{\left\{ \overline{\mu} \left[j \left(\gamma BB^\top P + BK_x \underline{\Gamma} \sigma \right) \right] - \omega_i \overline{\mu} \left[\left(BK_x + B\Theta_i^\top \Phi_i' \right) \right] \right\}}{\sqrt{\omega_i^2 \left\| BK_x + B\Theta_i^\top \Phi_i' \right\|^2 + \left\| \gamma BB^\top P + BK_x \underline{\Gamma} \sigma \right\|^2}} \right] \quad (121)$$

The frequency is computed from

$$0 \leq -\underline{\Gamma} \sigma + \overline{\mu} \left(A + B\Theta^{*\top} \Phi_i' \right) + \frac{1}{\omega_i} \overline{\mu} \left[-j \left(A + B\Theta^{*\top} \Phi_i' \right) \underline{\Gamma} \sigma \right] + \overline{\mu} \left[- \left(BK_x + B\Theta_i^\top \Phi_i' \right) \right] e^{-j\omega_i t_{d_i}}$$
$$+ \frac{1}{\omega_i} \overline{\mu} \left[j \left(\gamma BB^\top P + BK_x \underline{\Gamma} \sigma \right) \right] e^{-j\omega_i t_{d_i}}$$
$$\leq -\underline{\Gamma} \sigma + \overline{\mu} \left(A + B\Theta^{*\top} \Phi_i' \right) + \frac{1}{\omega_i} \overline{\mu} \left[-j \left(A + B\Theta^{*\top} \Phi_i' \right) \underline{\Gamma} \sigma \right] + \left\| BK_x + B\Theta_i^\top \Phi_i' \right\| + \left\| \gamma BB^\top P + BK_x \underline{\Gamma} \sigma \right\| \quad (122)$$

which yields the value of ω_i that renders t_d a minimum

$$\omega_i = \frac{\overline{\mu} \left[-j \left(A + B\Theta^{*\top} \Phi_i' \right) \underline{\Gamma} \sigma \right] + \left\| \gamma BB^\top P + BK_x \underline{\Gamma} \sigma \right\|}{\underline{\Gamma} \sigma - \overline{\mu} \left(A + B\Theta^{*\top} \Phi_i' \right) - \left\| BK_x + B\Theta_i^\top \Phi_i' \right\|} \quad (123)$$

provided $\underline{\Gamma} \sigma > \overline{\mu} \left(A + B\Theta^{*\top} \Phi_i' \right) + \left\| BK_x + B\Theta_i^\top \Phi_i' \right\|$.

Consider the case when $\Gamma \to \infty$. The time delay margin for the standard MRAC is known to be zero but remains finite for the σ-modification adaptive law. To see this, consider the time delay margin in the limit as $\Gamma \to \infty$ which becomes

$$t_{d_i} \leq \frac{1}{\omega_i} \cos^{-1} \left\{ \frac{\overline{\mu} \left[\left(A + B\Theta^{*\top} \Phi_i' \right) \right] + \overline{\mu} \left[j \left(\frac{\gamma BB^\top P}{\underline{\Gamma} \sigma} + BK_x \right) \right]}{\left\| \frac{\gamma BB^\top P}{\underline{\Gamma} \sigma} + BK_x \right\|} \right\} \quad (124)$$

$$\omega_i = \overline{\mu} \left[-j \left(A + B\Theta^{*\top} \Phi_i' \right) \right] + \left\| \frac{\gamma BB^\top P}{\underline{\Gamma} \sigma} + BK_x \right\| \quad (125)$$

If $\Gamma = cI$ where $c > 0$ is a constant, then

$$\frac{\gamma}{\underline{\Gamma} \sigma} = \frac{1}{\sigma T_0} \int_{t_i - T_0}^{t_i} \Phi^\top \left(x(\tau) \right) \Phi \left(x(\tau) \right) d\tau \quad (126)$$

is finite.

Thus, we conclude that t_{d_i} and ω_i also remain finite as $\Gamma \to \infty$. The σ-modification adaptive law is therefore robust. We can also see that as $\Gamma \to \infty$ in the limit, the error equation tends to

$$\dot{e}(t) = \left(A + B\Theta^{*\top} \Phi_i' \right) e(t) - \left(\frac{\gamma BB^\top P}{\underline{\Gamma} \sigma} + BK_x \right) e(t - t_d) - B\Theta^{*\top} \left[\Phi_i + \Phi_i' x_m(t) - \Phi_i' x_i \right]$$
$$- BK_x \left[x_m(t) - x_m(t - t_d) \right] + BK_r \left[r(t) - r(t - t_d) \right] \quad (127)$$

which yields the same time delay margin and frequency as in Eqs. (124) and (125).

It is noted that the frequency and time delay margin are dependent on time windows since γ varies with different time windows.

Example: For the same example in Section 4.2, the σ-modification adaptive law is implemented with $\sigma = 1$ and $\Gamma = 50$. Figure 5 is a plot of the time delay margin estimates using the matrix measure method and the numerical evidence of the time delay margin for both the σ-modification adaptive law and the standard MRAC. The time delay margin estimates are conservative but are fairly accurate. The estimation error is about 25% below the numerical evidence. The numerical evidence of the time delay margin for the standard MRAC is significantly below that for the σ-modification as expected due to lack of robustness in the standard MRAC.

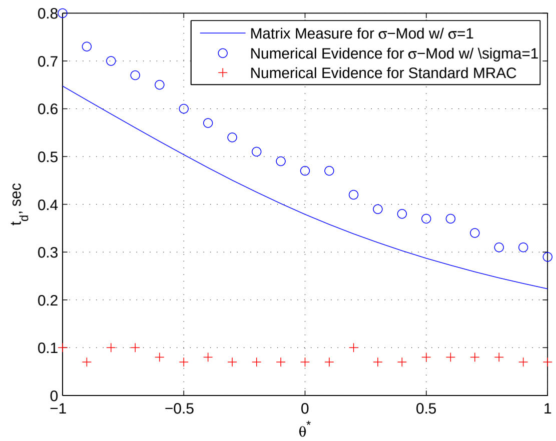

Fig. 5 - Time Delay Margin Estimate by Matrix Measure Method

4.4 Optimal Control Modification

The lack of robustness to unmodeled dynamics [16] and input time delay of the standard MRAC is well-known, as illustrated in the previous section. Increasing the adaptive gain allows the adaptation to attain a better tracking performance, but usually at the expense of the ability to maintain stability in the presence of unmodeled dynamics or time delay. To improve robustness, the two well-known robust modification methods; namely, the σ-modification [9] and ε-modification [17], have been used extensively in adaptive control. The optimal control modification method is a recently developed adaptive law to address robustness due to fast adaptation [10]. This adaptive law is given by

$$\dot{\Theta}(t) = -\Gamma\left[\Phi(x(t))\, e^{\top}(t)\, PB - \nu\Phi(x(t))\,\Phi^{\top}(x(t))\,\Theta(t)\, B^{\top} P A_m^{-1} B\right] \tag{128}$$

where $\nu > 0$ is a weighting constant.

In can be shown that in the limiting case as $\Gamma \to \infty$ and for a linear matched uncertainty where $\Phi(x(t)) = x(t)$, the optimal control modification exhibits an asymptotic linear behavior [19, 18]. This behavior can be explained using the analysis techniques developed herein.

The bounded linear stability analysis provides a method for analyzing an adaptive system with input time delay using a time windowing approach. Using the bounded linear approximation of an adaptive system, the matrix measure method can provide a non-conservative lower bound local estimate of the time delay margin. Unfortunately, the local estimation is non-unique and is dependent on the length of a time window, as is evident in Figure 2. The optimal control modification adaptive law has a unique property that enables it to be analyzed for stability without the use of the windowing approach of the bounded linear stability analysis as $\Gamma \to \infty$ for $\Phi(x(t)) = x(t)$. Thus, the time delay margin can be uniquely estimated for this adaptive law.

The bounded linear approximation of the optimal control modification adaptive law is

$$\dot{u}_{ad}(t) = -\gamma B^{\top} P e(t) + \gamma\nu B^{\top} A_m^{-\top} P B u_{ad}(t) + \Theta_i^{\top}\Phi_i'\left[\dot{x}_m(t) - \dot{e}(t)\right] \tag{129}$$

for $t \in [t_i - T_0, t_i)$, where $t_0 = 0$, $t_i = t_{i-1} + T_0$ and $i = 1, 2, \ldots, n \to \infty$.

Let $G = -BB^\top A_m^{-\top} P > 0$, then the error equation is obtained as

$$\ddot{e}(t) = \left(A + B\Theta^{*\top}\Phi_i' - \gamma\nu G\right)\dot{e}(t) - \left(BK_x + B\Theta_i^\top\Phi_i'\right)\dot{e}(t-t_d) + \gamma\nu G\left(A + B\Theta^{*\top}\Phi_i'\right)e(t)$$
$$- \left(\gamma BB^\top P + \gamma\nu GBK_x\right)e(t-t_d) - \gamma\nu GB\Theta^{*\top}\left[\Phi_i + \Phi_i' x_m(t) - \Phi_i' x_i\right] - \gamma\nu GBK_x\left[x_m(t) - x_m(t-t_d)\right]$$
$$+ \gamma\nu GBK_r\left[r(t) - r(t-t_d)\right] + B\Theta_i^\top\Phi_i'\dot{x}_m(t-t_d) - B\Theta^{*\top}\Phi_i'\dot{x}_m(t) - BK_x\left[\dot{x}_m(t) - \dot{x}_m(t-t_d)\right]$$
$$+ BK_r\left[\dot{r}(t) - \dot{r}(t-t_d)\right] \quad (130)$$

The time margin for the optimal control modification adaptive law can then be estimated by the following methods:

1. Padé Approximation:

   The time delay margin can be found from the following characteristic equation:

$$\det\left\{ j\omega_i I - \left(A + B\Theta^{*\top}\Phi_i'\right) + \left[BK_x + (j\omega_i I + \gamma\nu G)^{-1}\left(j\omega_i B\Theta_i^\top\Phi_i' + \gamma BB^\top P\right)\right]\frac{2 - j\omega_i t_{d_i}}{2 + j\omega_i t_{d_i}}\right\} = 0 \quad (131)$$

   or equivalently

$$\det\begin{bmatrix} j\omega_i I & -I & 0 \\ -\frac{2}{t_{d_i}}\left(A + B\Theta^{*\top}\Phi_i' - BK_x\right) + \gamma BB^\top P & j\omega_i I + \frac{2}{t_{d_i}} I - A - B\Theta^{*\top}\Phi_i' - BK_x - B\Theta_i^\top\Phi_i' & -\frac{2}{t_{d_i}} B - \frac{1}{t_{d_i}} B\gamma\nu G \\ -\gamma B^\top P & \Theta_i^\top\Phi_i' & j\omega_i I + \gamma\nu G \end{bmatrix} = 0 \quad (132)$$

2. Matrix measure method:

   Using the result for the σ-modification adaptive law, we obtain the time delay margin for the optimal control modification adaptive law as

$$t_{d_i} \le \frac{1}{\omega_i}\cos^{-1}\left[\left\{\omega_i^2 + \omega_i\overline{\mu}\left[j\left(A + B\Theta^{*\top}\Phi_i'\right)\right] + \overline{\mu}\left[\left(A + B\Theta^{*\top}\Phi_i'\right)\gamma\nu G\right]\right\} \times \right.$$
$$\times \frac{\left\{\left\|\gamma BB^\top P + BK_x\gamma\nu G\right\| - \omega_i\overline{\mu}\left[j\left(BK_x + B\Theta_i^\top\Phi_i'\right)\right]\right\}}{\omega_i^2\left\|BK_x + B\Theta_i^\top\Phi_i'\right\|^2 + \left\|\gamma BB^\top P + BK_x\gamma\nu G\right\|^2} +$$
$$\left. + \frac{\left\{\overline{\mu}\left[j\left(\gamma BB^\top P + BK_x\gamma\nu G\right)\right] - \omega_i\overline{\mu}\left[\left(BK_x + B\Theta_i^\top\Phi_i'\right)\right]\right\}}{\sqrt{\omega_i^2\left\|BK_x + B\Theta_i^\top\Phi_i'\right\|^2 + \left\|\gamma BB^\top P + BK_x\gamma\nu G\right\|^2}}\right] \quad (133)$$

$$\omega_i = \frac{\overline{\mu}\left[-j\left(A + B\Theta^{*\top}\Phi_i'\right)\gamma\nu G\right] + \left\|\gamma BB^\top P + BK_x\gamma\nu G\right\|}{\left\|\gamma\nu G\right\| - \overline{\mu}\left(A + B\Theta^{*\top}\Phi_i'\right) - \left\|BK_x + B\Theta_i^\top\Phi_i'\right\|} \quad (134)$$

   provided that $\|\gamma\nu G\| > \overline{\mu}\left(A + B\Theta^{*\top}\Phi_i'\right) + \left\|BK_x + B\Theta_i^\top\Phi_i'\right\|$.

   From the fact that $t_d \to 0$ as $\Gamma \to \infty$ for the standard MRAC, so a lower bound estimate of the time delay margin for which an adaptive law is stable can be estimated by the value of t_d that corresponds to $\Gamma \to \infty$ or equivalently $\gamma \to \infty$. Consider the case when $\gamma \to \infty$ and the matched uncertainty is linear with $\Phi(x(t)) = x(t)$. The asymptotic solution of the bounded linear approximation of the optimal control modification is obtained by taking the limit as $\gamma \to \infty$

$$Bu_{ad}(t) = \frac{1}{\nu}P^{-1}A_m^\top Pe(t) \quad (135)$$

   Then the asymptotic error equation as $\gamma \to \infty$ becomes

$$\dot{e}(t) = \left(A + B\Theta^{*\top}\right)e(t) - \left(BK_x - \frac{1}{\nu}P^{-1}A_m^\top P\right)e(t-t_d) - B\Theta^{*\top}x_m(t) - BK_x\left[x_m(t) - x_m(t-t_d)\right]$$
$$+ BK_r\left[r(t) - r(t-t_d)\right] \quad (136)$$

which is a LTI input delay equation independent of any time windowing parameters such as γ.

The characteristic equation of the asymptotic error equation can be obtained by letting $\gamma \to \infty$ which yields

$$\det\left[j\omega I - \left(A + B\Theta^{*\top}\right) + \left(BK_x + \frac{1}{\nu}G^{-1}BB^\top P\right)e^{-j\omega_i t_{d_i}}\right] = 0 \tag{137}$$

where $G^{-1}BB^\top P = -P^{-1}A_m^\top P$.

The time delay margin and frequency of the optimal control modification adaptive law as $\gamma \to \infty$ can then be estimated by the matrix measure method as

$$t_d \leq \frac{1}{\omega}\cos^{-1}\left[\frac{\overline{\mu}\left(A + B\Theta^{*\top}\right) + \overline{\mu}\left(j\left[BK_x - \frac{1}{\nu}P^{-1}A_m^\top P\right]\right)}{\left\|BK_x - \frac{1}{\nu}P^{-1}A_m^\top P\right\|}\right] \tag{138}$$

$$\omega = \overline{\mu}\left(-jA - jB\Theta^{*\top}\right) + \left\|BK_x - \frac{1}{\nu}P^{-1}A_m^\top P\right\| \tag{139}$$

The asymptotic results of t_d and ω can be verified to be the same as those from Eqs. (133) and (134) as $\gamma \to \infty$.

The effective phase margin is estimated as

$$\phi = \cos^{-1}\frac{\overline{\mu}\left(A + B\Theta^{*\top}\right) + \overline{\mu}\left(j\left[BK_x - \frac{1}{\nu}P^{-1}A_m^\top P\right]\right)}{\left\|BK_x - \frac{1}{\nu}P^{-1}A_m^\top P\right\|} \tag{140}$$

It is noted that both the asymptotic time delay and phase margins are independent of the time windows in the limit as $\Gamma \to \infty$ so the subscript i is dropped from the expressions. Because the time delay margin is a minimum as $\Gamma \to \infty$, the time delay margin estimate for the asymptotic solution of the optimal control modification adaptive law establishes a lower bound of the time delay margin for any adaptive gain $\Gamma < \infty$. Thus, to maintain stability of the input delay adaptive system using the optimal control modification adaptive law, a suitable selection of the weighting constant ν can be chosen to satisfy the time delay margin requirement and or phase margin requirement in an adaptive control design. In order to compute this estimate, the knowledge of the unknown parametric uncertainty Θ^* must be available.

Example: For the same example in Section 4.2, the optimal control modification (OCM) adaptive law is implemented with $\nu = 1$ and $\Gamma = 50$. Figure 6 is a plot of the time delay margin estimates using the matrix measure method and the numerical evidence of the time delay margin for both the optimal control modification adaptive law and the standard MRAC. The time delay margin estimates are reasonably conservative. The estimation error is greater for $\theta^* < 0$ but improves for $\theta^* > 0$. The numerical evidence of the time delay margin for the optimal control modification adaptive law is significantly greater than that for the standard MRAC. Therefore, the optimal control modification is robust.

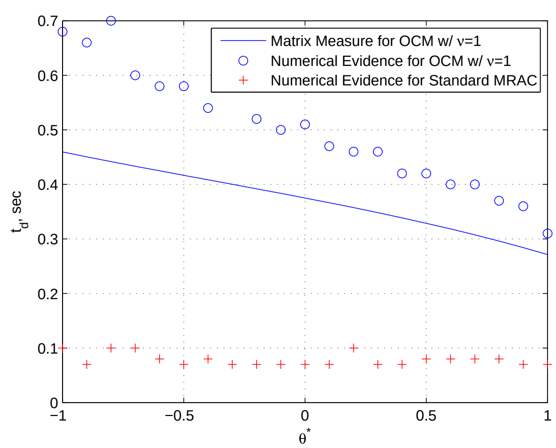

Fig. 6 - Time Delay Margin Estimate by Matrix Measure Method

5 Simulations

Consider a longitudinal pitch dynamic model of an aircraft

$$\begin{bmatrix} mV + \frac{C_{L_{\dot\alpha}}\bar{q}S\bar{c}}{2V} & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{C_{m_{\dot\alpha}}\bar{q}S\bar{c}^2}{2V} & 0 & I_{yy} \end{bmatrix} \begin{bmatrix} \dot\alpha(t) \\ \dot\theta(t) \\ \dot q(t) \end{bmatrix} = \begin{bmatrix} mg\gamma - C_{L_\alpha}\bar{q}S & -mg\gamma & mV - \frac{C_{L_q}\bar{q}S\bar{c}}{2V} \\ 0 & 0 & 1 \\ C_{m_\alpha} & 0 & \frac{C_{m_q}\bar{q}S\bar{c}^2}{2V} \end{bmatrix} \begin{bmatrix} \alpha(t) \\ \theta(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} -C_{L_{\delta_e}} \\ 0 \\ C_{m_{\delta_e}} \end{bmatrix} \left(\delta_e(t-t_d) + \begin{bmatrix} \theta^*_\alpha & 0 & \theta^*_q \end{bmatrix} \begin{bmatrix} \alpha(t) \\ \theta(t) \\ q(t) \end{bmatrix} \right) \quad (141)$$

where t_d is a time delay.

A numerical model for a full-scale generic transport model (GTM) at Mach 0.8 and 30,000 ft with the flight path angle $\gamma = 0$ is given by

$$\begin{bmatrix} \dot\alpha(t) \\ \dot\theta(t) \\ \dot q(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -0.7018 & 0 & 0.9761 \\ 0 & 0 & 1 \\ -2.6923 & 0 & -0.7322 \end{bmatrix}}_{A} \begin{bmatrix} \alpha(t) \\ \theta(t) \\ q(t) \end{bmatrix} + \underbrace{\begin{bmatrix} -0.0573 \\ 0 \\ -3.5352 \end{bmatrix}}_{B} \left(\delta_e(t-t_d) + \begin{bmatrix} 0.1393 & 0 & -0.2071 \end{bmatrix} \begin{bmatrix} \alpha(t) \\ \theta(t) \\ q(t) \end{bmatrix} \right)$$

A desired reference model of the pitch attitude is given by

$$\ddot\theta_m(t) + 2\zeta\omega_n\dot\theta_m(t) + \omega_n^2\theta_m(t) = \omega_n^2 r(t) \quad (142)$$

where $\zeta = 0.85$ and $\omega_n = 1.5$ rad/sec are chosen to give a desired handling characteristic.

Let $x(t) = \begin{bmatrix} \alpha(t) & \theta(t) & q(t) \end{bmatrix}^\top$, $u(t) = \delta_e(t)$, and $\Theta^{*\top} = \begin{bmatrix} \theta^*_\alpha & 0 & \theta^*_q \end{bmatrix} = \begin{bmatrix} 0.4 & 0 & -0.3071 \end{bmatrix}$. The parametric uncertainty Θ^* results in the short-period mode damping ratio of 0.095 which corresponds to almost neutral stability, whereas the nominal short-period mode has a damping ratio of 0.405. A nominal controller is designed as $u_{nom}(t) = -K_x x(t) + k_r r(t)$ where $K_x = \frac{1}{b_3}\begin{bmatrix} a_{31} & \omega_n^2 & 2\zeta\omega_n + a_{33} \end{bmatrix} = \begin{bmatrix} 0.7616 & -0.6365 & -0.5142 \end{bmatrix}$ and $k_r = \frac{1}{b_3}\omega_n^2 = -0.6365$. The closed-loop eigenvalues are -0.6582 and $-1.2750 \pm 0.7902i$. The nominal closed-loop plant is then chosen to be the reference model as

$$\underbrace{\begin{bmatrix} \dot\alpha_m(t) \\ \dot\theta_m(t) \\ \dot q_m(t) \end{bmatrix}}_{\dot x_m} = \underbrace{\begin{bmatrix} -0.6582 & -0.0365 & 0.9466 \\ 0 & 0 & 1 \\ 0 & -2.2500 & -2.5500 \end{bmatrix}}_{A_m} \underbrace{\begin{bmatrix} \alpha_m(t) \\ \theta_m(t) \\ q_m(t) \end{bmatrix}}_{x_m} + \underbrace{\begin{bmatrix} 0.0365 \\ 0 \\ 2.2500 \end{bmatrix}}_{B_m} r(t)$$

The control input is given by

$$u(t) = -K_x x(t) + k_r r(t) - \Theta^\top(t) x(t) \quad (143)$$

where $r(t)$ is a pitch attitude doublet.

Figures 7 and 8 are plots of the estimates of phase and time delay margins of the optimal control modification adaptive law for $\Gamma \to \infty$ computed by the matrix measure method from Eqs. (140) and (138) as a function of ν with and without the parametric uncertainty Θ^*. Note that the phase margin generally decreases as ν increases and reaches a steady state value, while the time delay margin reaches a maximum at about $\nu = 1$. Thus, for practical design purposes, ν should be kept between 0 and 1. A large value of ν produces a better time delay margin, but also results in a poorer tracking. For the specified uncertainty Θ^*, the maximum time delay margin is estimated to be 78 msec. Therefore, the input delay adaptive system will be stable with the optimal control modification adaptive law for any $t_d < 78$ msec.

Suppose the input time delay of the system is $t_d = 50$ msec. For this input time delay, the optimal control modification adaptive law produces a stable adaptation for $\nu \geq 0.244$. The controller is then implemented with $\nu = 0.25$ and $\Gamma = 3000I$ selected. Figures 9 and 10 illustrate the pitch angle and pitch rate responses due to the standard MRAC and optimal control modification adaptive law. The MRAC with the adaptive gain $\Gamma = 3000I$ does not track the reference pitch angle very well. High frequency oscillations are discernible in the pitch rate response. On the other hand, the optimal control modification adaptive law produces good tracking of the reference pitch angle and pitch rate. The high frequency oscillations in the pitch rate response with the standard MRAC is substantially reduced by the optimal control modification adaptive law.

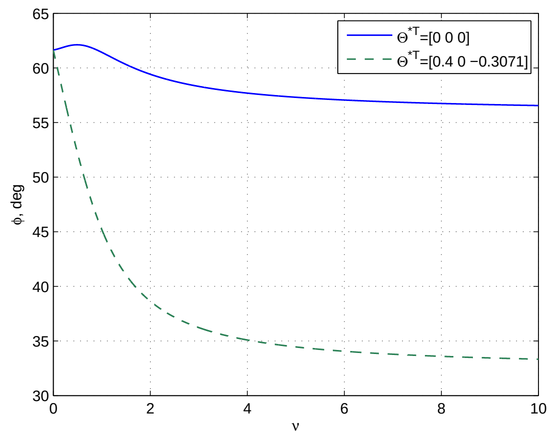

Fig. 7 - Phase Margin Estimation of Optimal Control Modification

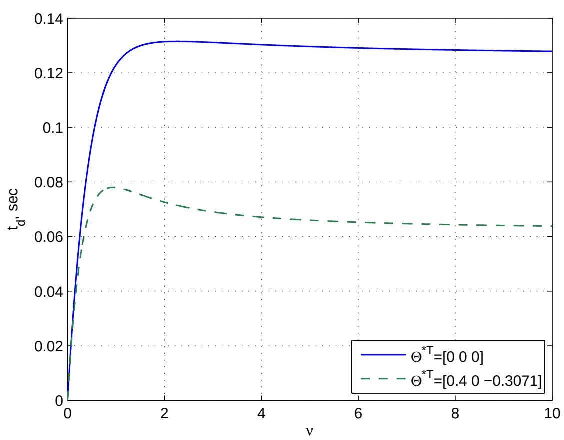

Fig. 8 - Time Delay Margin Estimation of Optimal Control Modification

The time delay margin of the closed-loop system is estimated by numerical evidence to verify the lower bound estimate of the time delay margin for the optimal control modification. The results are shown in Table 1.

	Time Delay Margin, msec
No Adaptation	550
MRAC	50
OCM	100

Table 1 - Time Delay Margin Estimates

The numerical evidence of the time delay margin for the optimal control modification adaptive law is estimated to be 100 msec. This is a factor of two larger than the time delay margin of 51 msec as estimated by the matrix measure method for $\nu = 0.25$.

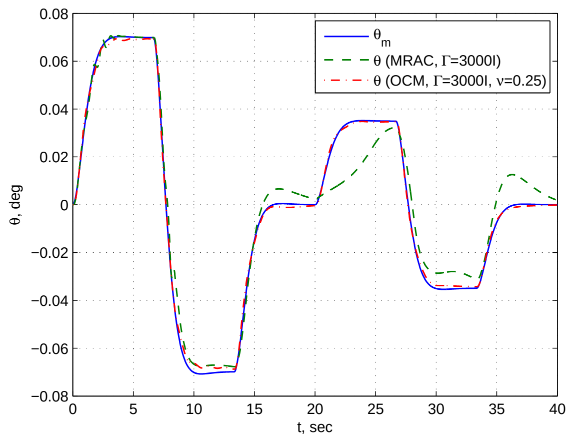

Fig. 9 - Pitch Angle Response

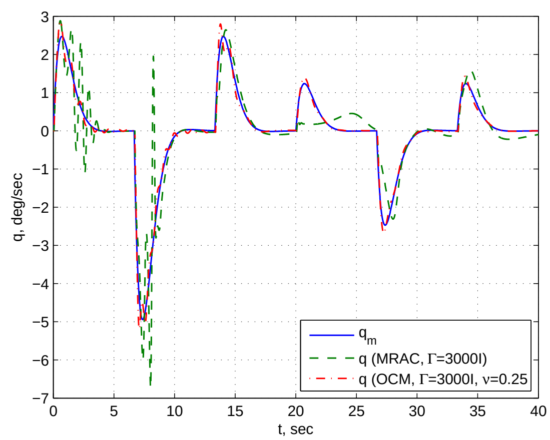

Fig. 10 - Pitch Rate Response

6 Conclusions

A method for analyzing input delay adaptive systems is presented. Three adaptive laws are considered in the study: standard MRAC, σ-modification, and optimal control modification. The bounded linear stability analysis approximates a nonlinear adaptive system as a bounded linear approximation within a time window. The windowing analysis is used to analyze local stability of the bounded linear approximation to estimate local stability behavior of the original nonlinear system within a given time window. Analytical approaches for computing the time delay margin are presented for three different methods: Padé approximation, Lyapunov-Krasovskii method with sum-of-squares optimization, and matrix measure method. Among the methods, the Padé approximation using a first-order Padé rational polynomial generally tends to be non-conservative, while the Lyapunov-Krasovskii method tends to be highly conservative in the time delay margin estimation. However, the sum-of-squares optimization demonstrates that a better Lyapunov-Krasovskii functional can be found by optimization to produce a less conservative time delay margin estimate. The matrix measure method seems to be able to estimate the time delay margin with reasonable accuracy. Moreover, the method is much simpler to use and does not require solving a linear matrix inequality as in the case of the Padé approximation or Lyapunov-Krasovskii method.

The asymptotic behavior of the time delay margin as the adaptive gain tends to infinity is studied. The standard model-reference adaptive control has zero time delay margin as the adaptive gain tends to infinity, as expected. The time delay margins for both the σ-modification adaptive law and optimal control modification adaptive law remain finite as the adaptive gain tends to infinity. The optimal control modification adaptive law also exhibits another useful feature in that the asymptotic value of the time delay margin is independent of the time window and the closed-loop input delay adaptive system tends to a LTI system. This behavior enables a lower bound of the time delay margin to be estimated with ease using the matrix measure method to guarantee stability for the input delay adaptive system. Flight control simulations demonstrate that the time delay margin estimation by the matrix measure method provides a good lower bound.

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