$\mathbf{L}_2$ gain verification for interconnections of locally stable systems using integral quadratic constraints

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Abstract— The paper establishes conditions which certify a local $\mathbf{L}_2$ gain for interconnections of locally stable systems. The approach involves improvements to local $\mathbf{L}_2$ gain estimation using reachability calculations. From these improved calculations, families of local integral quadratic constraints (IQCs) can be obtained for a fixed nonlinear dynamical system. Finally, the local IQCs are used to establish a local $\mathbf{L}_2$ gain using a state-space formulation of the IQC robust performance theorem.

I. INTRODUCTION

Integral quadratic constraints (IQCs), introduced in [8], provide a (possibly coarse) representation of nonlinearities and uncertainties that is useful for robustness analysis. Extensions to robust performance and synthesis have also proven effective [1], [6]. This paper presents an IQC-based analysis technique to analyze the input/output gain properties of interconnections of locally stable subsystems. We assume that each subsystem satisfies a collection of IQCs on inputs with $\mathbf{L}_2$ norm less than or equal to 1. The subsystems may be unstable (or even undefined) on larger norm input signals. The goal of analysis is to exploit these local IQCs to obtain a local bound on the gain of the interconnection.

We are interested in model-based certification of large-scale dynamical systems in the presence of input and model uncertainty. The complexity of such problems often dictates a decomposition approach, [13], breaking the system into a complex interconnection of smaller subsystems. Individual analysis on the isolated subsystems reveals coarse properties of the subsystems (eg., passive, small-gain or generalizations, such as dissipativeness with respect to various supply rates, etc.). In some cases, the coarse properties, coupled with the interconnection topology is enough to verify the overall behavior, [15], [7],[4]. In special cases, optimization can select which coarse properties are most important [13]. Moreover, this verification step, which involves the coarse properties and the interconnection topology is scalable, with semidefinite programming as the foundational computational engine. This paper extends these ideas to the situation where the decomposition leads to component models which are not globally stable, and may not even be defined on all inputs (eg., certain inputs may lead to finite-escape time solutions). We use IQCs as the generalizing notion of the coarse property concept.

As noted in [8], a more descriptive representation of a system described by IQCs can be potentially achieved if additional IQCs are known. Therefore, we will propose a method for constructing a list of IQCs for a system.

Our research relies on previous work in [3], [2], [12], [11], namely: Linear Matrix Inequalities (LMIs) [3], as a computationally-based analysis framework, which are widespread throughout system and control theory; formulations of the upper bounds on local $\mathbf{L}_2 \to \mathbf{L}_2$ input-output gains and reachable sets [2], [14] and [12]; and a power algorithm, [11], that attempts to find the worst case input that achieves the induced gain of a nonlinear system.

II. NOTATION

Let $\mathbb{R}$ denote the set of real numbers. A single superscript index denotes vectors, e.g. $\mathbb{R}^m$ is the set of $m \times 1$ vectors whose elements are in the $\mathbb{R}$.

$\mathbf{L}_2^m$ is the space of $\mathbb{R}^m$-valued functions $f : [0, \infty) \to \mathbb{R}^m$ of finite energy $\|f\|^2 = \int_0^\infty f(t)^T f(t)\, dt$. Functions that are square integrable on all finite intervals are notated $\mathbf{L}_{2e}^m$. Define $\|r\|_T^2 := \int_0^T r(t)^T r(t)\, dt$. Denote $\mathbf{L}_2^m[0, T]$ as the set of bounded energy functions defined on the domain $[0, T]$. For notational brevity, the superscript defining the dimensionality will not be specified throughout the paper. All signal norms are $\| \cdot \|_2$, although the subscript 2 is not shown.

An *operator* function is defined as $F : \mathbf{L}_{2e}^a \to \mathbf{L}_{2e}^b$. The *truncation* operator, P_T is defined as

$$(P_T f)(t) = \begin{cases} f(t) \text{ on } t \leq T \\ 0 \text{ on } t > T. \end{cases} \tag{1}$$

An operator, F, is *causal* if $P_T F = P_T F P_T$ for any $T > 0$. For LTI systems, no distinction is made between the operator and its transfer function.

The ball on $\mathbf{L}_2$ with radius R on is notated as $\mathbb{B}_R \mathbf{L}_2 := \{u \in \mathbf{L}_2 : \|u\| \leq R\}$.

For $\eta > 0$ and a function $g : \mathbb{R}^n \to \mathbb{R}$, define the η-sublevel set $\Omega_{g,\eta}$ of g as $\Omega_{g,\eta} := \{x \in \mathbb{R}^n : g(x) \leq \eta\}$.

For $\xi \in \mathbb{R}^n$, $\mathbb{R}[\xi]$ represents the set of polynomials in ξ with real coefficients. The subset $\Sigma[\xi] := \{\pi = \pi_1^2 + \pi_2^2 + \dots + \pi_M^2 : \pi_1, \dots, \pi_M \in \mathbb{R}[\xi]\}$ of $\mathbb{R}[\xi]$ is the set of sum-of-squares (SOS) polynomials.

III. INTEGRAL QUADRATIC CONSTRAINTS

Let $\Pi : j\mathbb{R} \to \mathbb{C}^{(l+m) \times (l+m)}$ be a measurable, bounded Hermitian-valued function. A bounded, causal operator Δ mapping $\mathbf{L}_2 \to \mathbf{L}_2$ is said to satisfy *the IQC defined by* Π, if for all $v \in \mathbf{L}_2$, with $w = \Delta(v)$, the inequality

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{v}(jw) \\ \widehat{w}(jw) \end{bmatrix}^* \Pi(jw) \begin{bmatrix} \widehat{v}(jw) \\ \widehat{w}(jw) \end{bmatrix} dw \geq 0 \tag{2}$$

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holds.

If Δ is not (or is not known to be) bounded (or even defined) on all of $\mathbf{L}_2$, then the notion of *locally* satisfying the IQC is relevant. Supposed $\Delta : \mathbb{B}_R \mathbf{L}_2 \to \mathbf{L}_2$ is causal and bounded. Then Δ is said to *locally satisfy the IQC by Π on R* if (2) holds for all $v \in \mathbb{B}_R \mathbf{L}_2$. Notice that Δ can be trivially extended to all of $\mathbf{L}_{2e}$ by defining

$$(\Delta_e(v))(t) = \begin{cases} (\Delta(v))(t) & \text{if } \|v\|_t \leq R \\ 0 & \|v\|_t > R \end{cases} \quad (3)$$

for $v \in \mathbf{L}_{2e}$. This operator, $\Delta_e : \mathbf{L}_{2e} \to \mathbf{L}_{2e}$, is causal, bounded and satisfies the IQC defined by Π on $\mathbb{B}_R \mathbf{L}_2$, but does not necessarily satisfy the IQC defined by Π on all of $\mathbf{L}_2$.

Time Domain

If Π is rational and uniformly bounded on the imaginary axis, then (2) can be expressed equivalently in the time-domain. By factorizing Π as $\Pi(jw) = \Psi^*(jw)M\Psi(jw)$, where M is a constant matrix, $\Psi(jw)$ is stable and proper, and $\Psi(jw) = C_\Psi(jwI - A_\Psi)^{-1}[B_{1,\Psi} \ B_{2,\Psi}] + [D_{1,\Psi} \ D_{2,\Psi}]$, the IQC in (2) is expressed as:

$$\int_0^\infty y_\Psi^T(t) M y_\Psi(t) \, dt \geq 0, \quad (4)$$

where y_Ψ is the output of

$$x_\Psi(0) = 0 \quad (5)$$
$$\dot{x}_\Psi = A_\Psi x_\Psi(t) + B_{1,\Psi} v(t) + B_{2,\Psi} w(t) \quad (6)$$
$$y_\Psi = C_\Psi x_\Psi(t) + D_{1,\Psi} v(t) + D_{2,\Psi} w(t). \quad (7)$$

If

$$\int_0^T y_\Psi^T(t) M y_\Psi(t) \, dt \geq 0, \quad (8)$$

for all T, then Δ is said to satisfy the *hard* IQC defined by Π. The fact that the sign of the integral in (8) is known at intermediate times will be used in section IV. If only (4) holds, then Δ is said to satisfy the *soft* IQC defined by Π.

IV. LOCAL INPUT/OUTPUT GAIN USING IQCS

Consider the system in Figure 1, which is specified by the equations

$$\begin{bmatrix} z \\ e \end{bmatrix} = G \begin{bmatrix} w \\ d \end{bmatrix}, \quad (9)$$
$$w_i = \Delta_i(z_i), i = 1, \ldots, N \quad (10)$$

where G is a causal, finite dimensional, linear time-invariant operator on $\mathbf{L}_{2e}$ and the subscript i denotes the i'th component of a partitioned vector or diagonal system.

The operators $\Delta_i : \mathbb{B}_1 \mathbf{L}_2 \to \mathbf{L}_2$ are bounded and causal (but not necessarily defined on all of $\mathbf{L}_2$). Moreover, each Δ_i locally, on $\mathbb{B}_1 \mathbf{L}_2$, satisfies the hard IQCs defined by $\{\Pi_{ij}\}_{j=1}^M$ and

$$\Pi_{ij}(jw) = \Psi_{ij}^*(jw) \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \Psi_{ij}(jw) \quad (11)$$
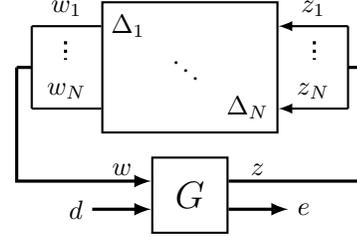$$\forall i = 1, \ldots, N, \forall j = 1, \ldots, M.$$

Fig. 1. Feedback Interconnection of (G, Δ)

The $\Psi_{ij}(s)$ are stable, finite dimensional linear systems with

$$\dot{x}_{\Psi_{ij}} = A_{ij} x_{\Psi_{ij}} + B_{1,ij} z_i + B_{2,ij} w_i \quad (12)$$
$$\begin{bmatrix} p_{ij} \\ q_{ij} \end{bmatrix} = C_{ij} x_{\Psi_{ij}} + D_{1,ij} z_i + D_{2,ij} w_i. \quad (13)$$

Let Δ denote the block diagonal concatenation of the Δ_i operators. Similar to (3), each Δ_i can be trivially extended to all of $\mathbf{L}_{2e}$ by defining

$$(\Delta_{e,i}(v))(t) = \begin{cases} (\Delta_i(v))(t) & \text{if } \|v\|_t \leq 1, \\ 0 & \text{if } \|v\|_t > 1. \end{cases} \quad (14)$$

We wish to find a bound on the $\mathbf{L}_2 \to \mathbf{L}_2$ gain from exogenous input d to the output e, in Figure 1, which is valid under some (unknown at this point) bound on $\|d\|$. The analysis will be accomplished by using the information about each Δ_i, contained in the IQCs defined by $\{\Pi_{ij}\}_{j=1}^M$ in (11).

The advantage of using IQCs is that the knowledge about the input-output behavior of Δ_i is encapsulated within (11). Therefore the Δ_i systems are removed from the analysis, since only the IQCs are needed, which is represented in Figure 2, where w is viewed as an external signal.

In the proof, we will first investigate the gain from d to any one of the components of z, z_k, by temporarily setting $e := z_k$, which yields the structure shown in Figure 3. The system G_k embodies this error redefinition.

Theorem 1: Let $G(s)$ be a causal, finite dimensional, linear time-invariant operator and let $\{\Delta_i\}_{i=1}^N$ be causal, bounded operators mapping $\mathbb{B}_1 \mathbf{L}_2 \to \mathbf{L}_2$. For each i, Δ_i locally (on $\mathbb{B}_1 \mathbf{L}_2$) satisfies the IQCs defined by $\{\Pi_{ij}\}_{j=1}^M$. Each Π_{ij} is represented by a linear system, as in (12)-(13). Let x be the state of G, and x_Ψ be the concatenated state of all Ψ_{ij}. Finally, let $\Delta_{e,i}$ denote the extension introduced in (14). For notational simplicity, define $\Delta := \text{diag}\{\Delta_i\}$ as the block diagonal concatenation. Similarly, $\Delta_e := \text{diag}\{\Delta_{e,i}\}$.

Assume $\alpha > 0, \beta > 0$ and

1) the interconnection of G and Δ_e, shown in Figure 4, is well-posed;
2) for each $1 \leq k \leq N$ there exist positive semidefinite quadratic function $V_k(x, x_\Psi) = \begin{bmatrix} x \\ x_\Psi \end{bmatrix}^* P_k \begin{bmatrix} x \\ x_\Psi \end{bmatrix}$ and $\{\lambda_{ijk}\} \geq 0$ such that the linear system shown in Figure
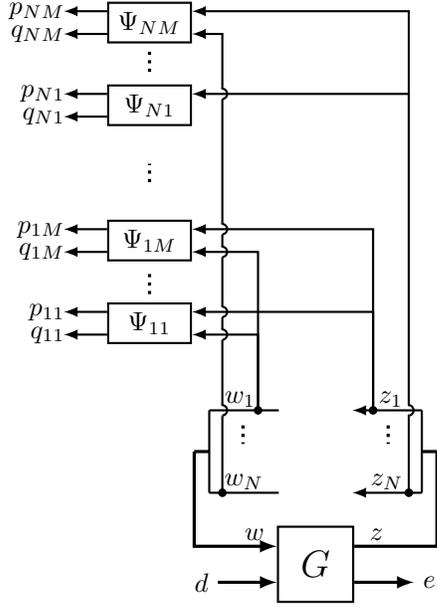
Fig. 2. *
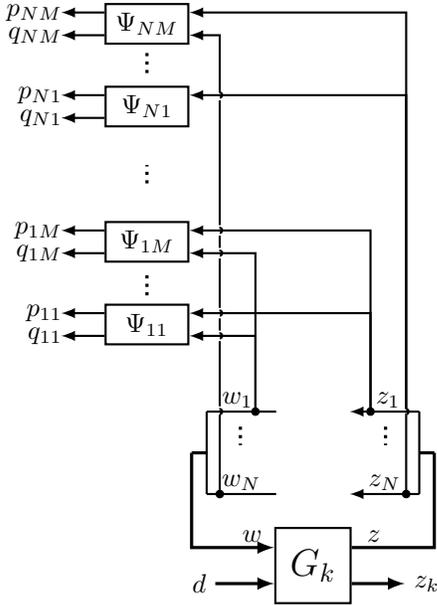Analysis Interconnection

Fig. 3. *
Analysis Interconnection
with $e := z_k$

3 satisfies

$$\dot{V}_k \leq \frac{1}{\beta^2} d^T d - z_k^T z_k + \sum_{i=1}^{N} \sum_{j=1}^{M} \lambda_{ijk}(q_{ij}^T q_{ij} - p_{ij}^T p_{ij}).$$

(15)

3) there exist positive semidefinite quadratic function $V_0(x, x_\Psi) = \begin{bmatrix} x \\ x_\Psi \end{bmatrix}^* P_0 \begin{bmatrix} x \\ x_\Psi \end{bmatrix}$ and $\lambda_{ij0} \geq 0$ such that

the linear system shown in Figure 2 satisfies

$$\dot{V}_0 \leq \alpha^2 d^T d - e^T e + \sum_{i=1}^{N} \sum_{j=1}^{M} \lambda_{ij0}(q_{ij}^T q_{ij} - p_{ij}^T p_{ij}).$$

(16)

Then, the feedback interconnection of (G, Δ) is well-posed for all $d \in \mathbf{L}_2$ with $\|d\| < \beta$. Moreover, each z_k satisfies $\|z_k\| \leq 1$ and $\|e\| \leq \alpha\|d\|$.

Remark 1: The inequalities in (15) and (16) are quadratic constraints on the variables (x, x_Ψ, d, w), parameterized by P_k, P_0, λ_{ijk}, and λ_{ij0}. Hence (15) and (16) are LMIs, [16], [3], in P and λ. Moreover, these state-space conditions can be rewritten as a frequency-domain conditions as in [9], [10], [16], eliminating explicit computation of the "storage" functions V_k and V_0. In the frequency-domain formulation, only the λ_{ijk} and λ_{0ij} are decision variables, giving a frequency-dependent semidefinite constraint.

Proof: The proof is given for $\alpha = \beta = 1$. The interconnection of interest is (G, Δ). However, we initially quantify the behavior of the well-posed interconnection (G, Δ_e), shown in Figure 4 and, at the end of the proof, relate the solutions of (G, Δ_e) to (G, Δ). Let $d \in \mathbf{L}_{2e}$, $\|d\| < 1$. Since $d \in \mathbf{L}_{2e}$, unique solutions in $\mathbf{L}_{2e}$ exist. Suppose there exists a k such that $\|z_k\|_{\bar{T}} > 1$ at some $\bar{T} > 0$. Since $z_k \in \mathbf{L}_{2e}$, $\|z_k\|_T$ is a continuous, nondecreasing, function of T and is equal to 0 at $T = 0$. Therefore, at some $T_1 < \bar{T}$, there exists an index m (possibly equal to k) such that $\|z_m\|_{T_1} = 1$ and $\|z_i\|_{T_1} \leq 1$ for all $i \neq m$. Note that for all $T_2 \leq T_1$ and all i, $\|z_i\|_{T_2} \leq 1$. Therefore on the time interval $[0, T_2]$, the hard IQCs for each $\Delta_{e,i}$ are satisfied. Hence for all i, j

$$\|q_{ij}\|_{T_2} \leq \|p_{ij}\|_{T_2}.$$

(17)

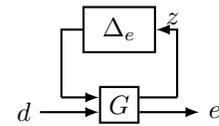From well-posedness of the interconnection of (G, Δ_e),

Fig. 4.   Feedback Interconnection of (G, Δ_e)

we can integrate (15) with the initial condition $x(0) = 0$ and $x_\Psi(0) = 0$, yielding for all k

$$\|z_k\|_{T_2}^2 + V_k(x(T_2), x_\Psi(T_2)) \leq \|d\|_{T_2}^2$$
$$+ \sum_{i=1}^{N} \sum_{j=1}^{M} \lambda_{ijk}(\|q_{ij}\|_{T_2}^2 - \|p_{ij}\|_{T_2}^2).$$

(18)

From the positive semidefiniteness of V_k and the hard IQC conditions in (17)

$$\|z_k\|_{T_2}^2 \leq \|d\|_{T_2}^2$$

(19)

holds for all k. However, with $k = m$, we know $\|z_k\|_{T_2}^2 = 1$, which contradicts $\|d\| < 1$.

The proof of $\|e\| \leq \|d\|$ follows similarly by integrating (16).

Summarizing, for (G, Δ_e), we have shown that $\|d\| < 1$ implies all $\|z_k\| \leq 1$ and $\|e\| \leq \|d\|$. However, we are ultimately interested in the interconnection of (G, Δ), in Figure 1. Since $\Delta_{e,i}|_{\mathbb{B}_1 \mathbf{L}_2} = \Delta_i|_{\mathbb{B}_1 \mathbf{L}_2}$, any fact about the solutions of (G, Δ_e), which satisfies $\|z_k\| \leq 1$ for all k, is also true for (G, Δ). ∎

The potential usefulness of this theorem critically relies on the existence and availability of local IQCs for the various Δ_i operators. In the next few sections, V, VI, and VII, some methods toward obtaining local IQCs are presented. It is anticipated that these methods, or other techniques, will continue to advance, leading to reliable and automatic generation of high quality, locally satisfied IQCs for a given nonlinear system.

V. ESTABLISHING LOCAL, HARD IQCS

We outline a procedure to generate locally satisfied IQCs for a nonlinear dynamical system, using linear offsets, linear weighting functions and estimates of local $\mathbf{L}_2$ gains.

Theorem 2: If Δ is a bounded, causal operator mapping $\mathbb{B}_R \mathbf{L}_2 \to \mathbf{L}_2$, and Q and W are linear, time-invariant, stable, then $W(\Delta - Q)$ is bounded and causal on $\mathbb{B}_R \mathbf{L}_2$. Furthermore, if $\|W(\Delta - Q)\| \leq 1$ on $\mathbb{B}_R \mathbf{L}_2$, then Δ locally satisfies the IQC defined by Π, where $S(jw) := W(jw)Q(jw)$ and

$$\Pi(jw) = \begin{bmatrix} I - S^*(jw)S(jw) & S^*(jw)W(jw) \\ W^*(jw)S(jw) & -W^*(jw)W(jw) \end{bmatrix}. \quad (20)$$

Proof: For $v \in \mathbb{B}_R \mathbf{L}_2$, define $z := W(\Delta(v) - Qv)$, as shown in Figure 5. Clearly $z \in \mathbf{L}_2$ and $\|z\| \leq \|v\|$ by assumption on the local $\mathbf{L}_2$ gain of $W(\Delta - Q)$. In terms of Fourier transforms,

$$\begin{bmatrix} \widehat{v}(jw) \\ \widehat{z}(jw) \end{bmatrix} = \begin{bmatrix} I & 0 \\ -S(jw) & W(jw) \end{bmatrix} \begin{bmatrix} \widehat{v}(jw) \\ \widehat{\Delta(v)}(jw) \end{bmatrix}. \quad (21)$$

By Parseval's theorem, $\|z\| \leq \|v\|$ is equivalent to

$$\int_R \begin{bmatrix} \widehat{v}(jw) \\ \widehat{z}(jw) \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \widehat{v}(jw) \\ \widehat{z}(jw) \end{bmatrix} dw \geq 0. \quad (22)$$

Direct substitution of (21) into (22) yields

$$\int_R \begin{bmatrix} \widehat{v}(jw) \\ \widehat{\Delta(v)}(jw) \end{bmatrix}^* \Pi(jw) \begin{bmatrix} \widehat{v}(jw) \\ \widehat{\Delta(v)}(jw) \end{bmatrix} dw \geq 0 \quad (23)$$

as desired, factored as

$$\Psi(jw) = \begin{bmatrix} I & 0 \\ -S(jw) & W(jw) \end{bmatrix}, M = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

∎
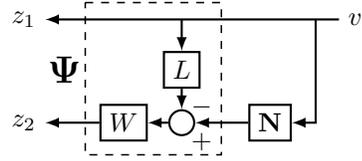

Fig. 5.   IQC Interconnection

VI. LOCAL $\mathbf{L}_2 \to \mathbf{L}_2$ GAIN

In this section, we discuss dissipation inequalities, [17], [5], [14], [2], for local $\mathbf{L}_2 \to \mathbf{L}_2$ gain bounds. Local $\mathbf{L}_2 \to \mathbf{L}_2$ gain bounds are a critical component of section V, which establishes more general local IQCs.

Consider an affine, nonlinear system, with $x(0) = 0$ defining an operator $\mathbf{N}$ mapping u to y as

$$\dot{x} = f(x) + g(x)u =: F(x, u), \quad (24)$$

$$y = h(x) + k(x)u =: H(x, u). \quad (25)$$

If the dynamical system $\mathbf{N}$ satisfies $\|y\| \leq \gamma \|u\|$ for all $\|u\| \leq R$, then we notate this as $\|\mathbf{N}\| \leq \gamma$ on $\mathbb{B}_R \mathbf{L}_2$.

Theorem 3: If there exists a continuously differentiable $V : \mathbb{R}^n \to \mathbb{R}$, $Q : \mathbb{R}^n \to \mathbb{R}$, $R \geq 0$ and $\gamma > 0$ such that

$$V(0) = 0, Q(0) = 0, V(x) > 0, Q(x) > 0 \; \forall x \neq 0, \quad (26)$$

$$\Omega_{Q,\tau^2} \subseteq \Omega_{V,R^2} \text{ is bounded}, \quad (27)$$

$$\nabla V \cdot F(x, u) \leq u^T u - \frac{1}{\gamma^2} H^T H \quad (28)$$

$$\forall x \in \Omega_{V,R^2} \text{ and } \forall u \in \mathbb{R}^m, \text{ and}$$

$$\nabla Q \cdot F(x, u) \leq u^T u \quad \forall x \in \Omega_{Q,\tau^2} \text{ and } \forall u \in \mathbb{R}^m, \quad (29)$$

then $\|\mathbf{N}\| \leq \gamma$ on $\mathbb{B}_\tau \mathbf{L}_2$.

Proof: This is easily derived using the reachability results in [14]. ∎

Remark 2: It is important to note that the operator $\mathbf{N}$, as governed by (24) and (25), may not be defined on inputs outside of $\mathbb{B}_R \mathbf{L}_2$.

VII. SOS CONDITIONS FOR LOCAL $\mathbf{L}_2 \to \mathbf{L}_2$ GAIN

In this section we review some computational methods, to verify the conditions in section VI. The methods are applicable to polynomial systems and employ SOS programming.

If $f(x)$, $g(x)$, $h(x)$ and $k(x)$ are polynomials and $V(x)$ is restricted to be a polynomial, then the S-procedure and SOS conditions can verify (26)-(28). Specifically, for given $\gamma > 0$, $R > 0$, and positive definite polynomial l, the conditions

$$s_1 \in \Sigma[x, u], V(0) = 0, \quad (30)$$

$$V - l \in \Sigma[x], \quad (31)$$

$$u^T u - \frac{1}{\gamma^2} y^T y - \nabla V F(x, u) - s_1(R^2 - V) \in \Sigma[x, u] \tag{32}$$

are sufficient for (26)-(28). For fixed γ, maximizing R by choice of V, R, and s_1 is natural, but results in a bilinear and nonconvex optimization detailed in [2].

Restricting $Q(x)$ to be a polynomial, for a given $\tau > 0$, the conditions

$$s_2 \in \Sigma[x, u], s_3 \in \Sigma[x], Q(0) = 0, Q \in \Sigma[x], \tag{33}$$

$$u^T u - \nabla Q \cdot F(x, u) - s_2(\tau^2 - Q) \in \Sigma[x, u], \tag{34}$$

$$R^2 - V - s_3(\tau^2 - Q) \in \Sigma[x] \tag{35}$$

satisfy (27)-(29). For any feasible V and R in (30)-(32), $\tau := R$ and $Q := V$ are feasible for (33)-(35). However, maximizing τ (the bound on u) by choice of Q, τ, s_2, and s_3 for (33)- (35) is the ultimate goal.

Remark 3: Note that the refinement procedure in [14], which improves the estimates using a polynomial storage function by a special transformation to a non-polynomial storage function, can be applied to equations (34) and (35).

VIII. EXAMPLE

We conclude with a toy example to illustrate the ideas. A 1-state nonlinear system $\mathbf{N}$ is $\dot{x} = -x + x^3 + u$, $y = x$, which is locally exponentially stable, but can exhibit finite-escape time solutions if $\|u\|_2 > 1$. We establish simple local IQCs for this system using the approach in section V. Specifically, choose a linear offset $Q(s) := \frac{1}{s+1}$, which is just the linearization of $\mathbf{N}$. Next, following the procedure in sections VI and VII, estimate the induced $\mathbf{L}_2 \to \mathbf{L}_2$ gain of the locally stable operator $\mathbf{N} - Q(s)$. The gain, which depends on the norm-bound of the input is 0 for arbitrarily small inputs, and goes to ∞ as the norm of the input is allowed to approach 1. Bounds on the gain, as a function of input-norm level, are shown below in Figure 6. The upper (red) curve is an upper bound on the gain, obtained with the SOS analysis suite, [2], using storage functions V and Q of degree 6, and the lower (blue, dashed) curve is a lower bound on the gain, obtained using simulation-based techniques introduced in [11].

The horizontal axis is labeled R, using the notation of section V. Each point on the upper curve gives rise to a local IQC, with $Q(s)$ as defined, R as the horizontal coordinate, and $W(s)$ being the reciprocal of the norm bound (ie., the reciprocal of the vertical component of the point).

In order to illustrate the results of section IV, we consider a simple interconnection of 3 copies of $\mathbf{N}$, shown in Figure 7 along with a disturbance d and error e, where $\Gamma := 1.05$.

Using a direct approach (eg., the methods of sections VI, VII, for comparison purposes only, possible due to the small size of this interconnection, but not the point of this paper), a bound on the $\mathbf{L}_2$ gain from $d \to e$ can be calculated. For example, Figure 8 shows the gain bound achieved using quadratic (square points) and quartic (asterisk points) V (in
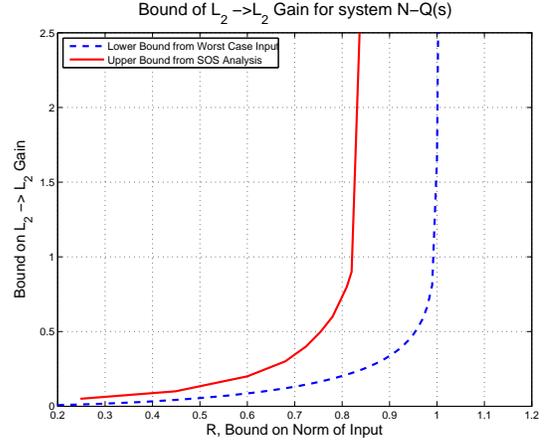

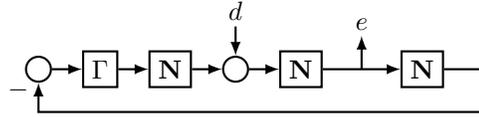Fig. 6. Bound of $\mathbf{L}_2 \to \mathbf{L}_2$ Gain for system $N - Q$

Fig. 7. Feedback System

each case, the degree of Q is chosen to equal the degree of V).

Alternatively, one can select three points (since there are 3 $\mathbf{N}$ blocks) from the gain curve for $\mathbf{N}$ in Figure 6 and use the associated IQCs, and the analysis in section IV to bound the gain, using only the IQC information of $\mathbf{N}$, and not $\mathbf{N}$ itself. The semidefinite programs, (15) and (16), run very quickly (fractions of a second on basic desktop) yielding the gain bound guarantee in Figure 8 (diamond curve). While this is clearly suboptimal to the direct answer, the methods in this paragraph should be scalable to include hundreds of subsystems. Future work will involve analysis with multiple IQCs for each $\mathbf{N}$ block.
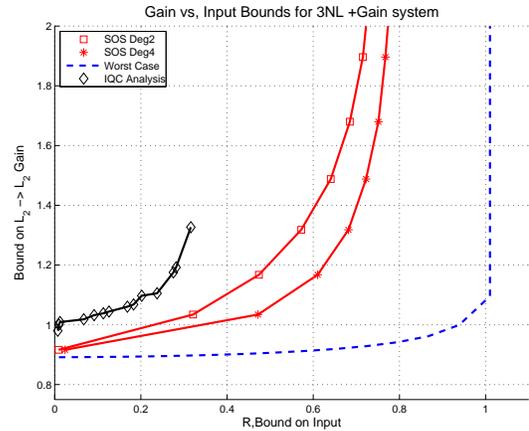

Fig. 8. Gain Bounds vs. Input Bounds for SOS, IQC and Worst-Case analysis

IX. Future Work

Ultimately, we envision the possibility of creating large libraries (with hundreds of thousands of entries) of small (1-4 state) nonlinear system models, each with an associated (long) list of IQCs which the model satisfies (locally). A large system can be decomposed into an interconnection, and the interconnection can be quickly analyzed, using the library. If the analysis is inconclusive, an alternate decomposition can be proposed, and the analysis repeated. There are several issues to be worked out, but the idea of huge, machine-searchable libraries of small, dynamic-model building-blocks, and their coarse, analysis-oriented properties is at the heart of this paper.

X. Conclusions

A method for verifying the $\mathbf{L}_2$ gain for interconnections of locally stable systems using IQCs was presented. This relies on the availability of local IQCs. We also presented ideas for constructing local IQCs for fixed nonlinear systems using SOS techniques.

XI. Acknowledgments

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